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Mixed-mode oscillations in a stochastic, piecewise-linear system

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ABSTRACT

We analyse a piecewise-linear FitzHugh–Nagumo model. The system exhibits a canard near which both small amplitude and large amplitude periodic orbits exist. The addition of small noise induces mixed-mode oscillations (MMOs) in the vicinity of the canard point. We determine the effect of each model parameter on the stochastically driven MMOs. In particular we show that any parameter variation (such as a modification of the piecewise-linear function in the model) that leaves the ratio of noise amplitude to time-scale separation unchanged typically has little effect on the width of the interval of the primary bifurcation parameter over which MMOs occur. In that sense, the MMOs are robust. Furthermore, we show that the piecewise-linear model exhibits MMOs more readily than the classical FitzHugh–Nagumo model for which a cubic polynomial is the only nonlinearity. By studying a piecewise-linear model, we are able to explain results using analytical expressions and compare these with numerical investigations.

1. Introduction

Oscillatory dynamics involving oscillations with greatly differing amplitudes, known as mixed-mode oscillations (MMOs), see Fig. 1, are important in neuron models [1] and in a multitude of chemical reactions [2,3]. Yet there are many open questions regarding the creation, robustness and bifurcations of MMOs. A variety of mechanisms generate MMOs in deterministic systems. These include the existence of a Shil'nikov-type homoclinic orbit or certain heteroclinic connections, folds on a slow manifold of a slow–fast system, and a subcritical Hopf bifurcation; see [4] and references within. Alternatively MMOs may be noise-induced; we discuss some scenarios by which this may occur below.

Oscillations are fundamental to the FitzHugh–Nagumo (FHN) model – dating from the early 1960s [5,6] – that is used as a prototypical model of excitable dynamics in a range of scientific fields [7,8]. We study the following form of the FHN model with small, additive, white noise:

$$\begin{aligned} dv &= (f(v) - w)dt, \\ dw &= \varepsilon(\alpha v - \sigma w - \lambda)dt + D dW, \end{aligned} \tag{1}$$

where v represents a potential, w is a recovery variable and W is a standard Brownian motion. Here α is a positive constant and $\lambda \in \mathbb{R}$, which is regarded as the main bifurcation parameter, controls the growth of oscillations, as seen below. The small parameter $\varepsilon \ll 1$, represents the time-scale separation and $D \ll 1$ is the noise amplitude ($\varepsilon, D > 0$). Values of ε and D used in, for instance [9,10], are not larger than the values considered here. By scaling we may assume $\sigma = 1$, except in the special case $\sigma = 0$ which corresponds, in the absence of noise, to the van der Pol model (and in this case we may further assume $\alpha = 1$). We assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and S-shaped in that f has one local minimum and one local maximum. For simplicity, we assume that the local minimum is at (0, 0) and the local maximum is at (1, 1), regardless of the precise function chosen.

If f is a cubic, as originally taken by FitzHugh [5] and Nagumo et al. [6], then, by the above requirements, the cubic must be

$$f(v) = 3v^2 - 2v^3. \tag{2}$$

Fig. 2(A) illustrates the role of the parameter λ for (1) with (2) in the absence of noise. A small amplitude periodic orbit is created in a Hopf bifurcation at $\lambda = 0$. For the parameters used in Fig. 2, this periodic orbit is stable and its amplitude increases with λ. Near λ_c the amplitude increases to order one over a parameter range that is exponentially small in ε. This rapid growth is known as a canard explosion and is due to time-scale separation and global dynamics [11–14]. The value of the *canard point*, λ_c, which is well defined for smooth systems [15,16], decreases to zero with ε, as shown in Fig. 3(A). Over an order ε range of λ values, (1) with (2) may either settle to equilibrium, exhibit small amplitude oscillations, or exhibit large amplitude oscillations (the latter are also relaxation oscillations).

As in [17,18], here we study a piecewise-linear (PWL) FHN model so that, in the presence of noise, the system is amenable to a rigorous treatment without the need for an asymptotic analysis in

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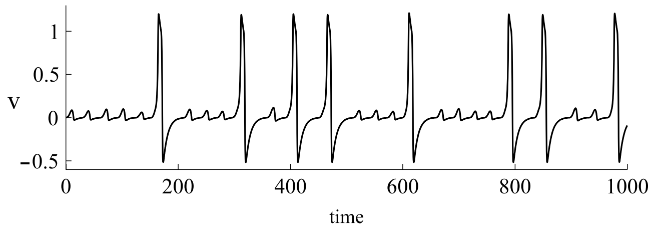

Fig. 1. A time series illustrating MMOs exhibited by (1) with (3). For this plot, $\lambda = 0.028, D = 0.0008, \varepsilon = 0.04, (\alpha, \sigma) = (4, 1), (\eta_L, \eta_R) = (-2, -1)$ and $(v_1, w_1) = (0.1, 0.05)$.

the time-scale separation parameter, ε. PWL models are commonly used in circuit systems [19–21]. A PWL version of a driven van der Pol oscillator is studied in [22] to explain the breakdown of canards in experiments. We consider the continuous, PWL function

$$f(v) = \begin{cases} \eta_L v, & v \le 0 \\ \eta_1 v, & 0 < v \le v_1 \\ \eta_2 (v - v_1) + w_1, & v_1 < v \le 1 \\ \eta_R (v - 1) + 1, & v > 1 \end{cases}, \qquad (3)$$

where $0 < v_1, w_1 < 1, \eta_L, \eta_R < 0$, and

$$\eta_1 = \frac{w_1}{v_1}, \qquad \eta_2 = \frac{1 - w_1}{1 - v_1}. \qquad (4)$$

The PWL function (3) is pictured below in Fig. 5. We state it here in order to briefly illustrate key differences between the smooth and PWL FHN models. Further motivation for the particular form (3) is given in Section 2.

As shown in Fig. 2(B), (1) with (3) may exhibit a canard explosion. The canard point, λ_c, is not well defined for this system because it lacks global differentiability. Instead we consider the values, λ_{v_1} and λ_1, at which the maximum v-value of the periodic

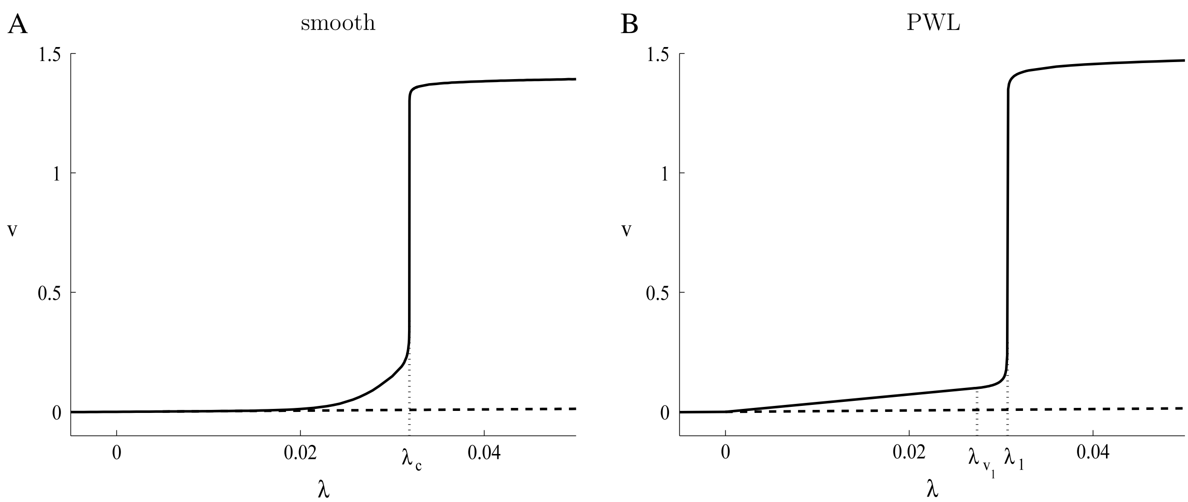

Fig. 2. Bifurcation diagrams of (1) in the absence of noise (i.e. $D = 0$) with (2) in panel A and with (3) in panel B. In each panel the solid curve for $\lambda > 0$ corresponds to the maximum v-value of a stable periodic orbit; the remaining curves correspond to the equilibrium which is unstable for $\lambda > 0$. In panel A a canard explosion occurs near the canard point, λ_c; in panel B a canard explosion occurs near λ_1 at which point the stable periodic orbit has a maximum value of 1. The parameter values used are the same as in Fig. 1, except here $D = 0$.

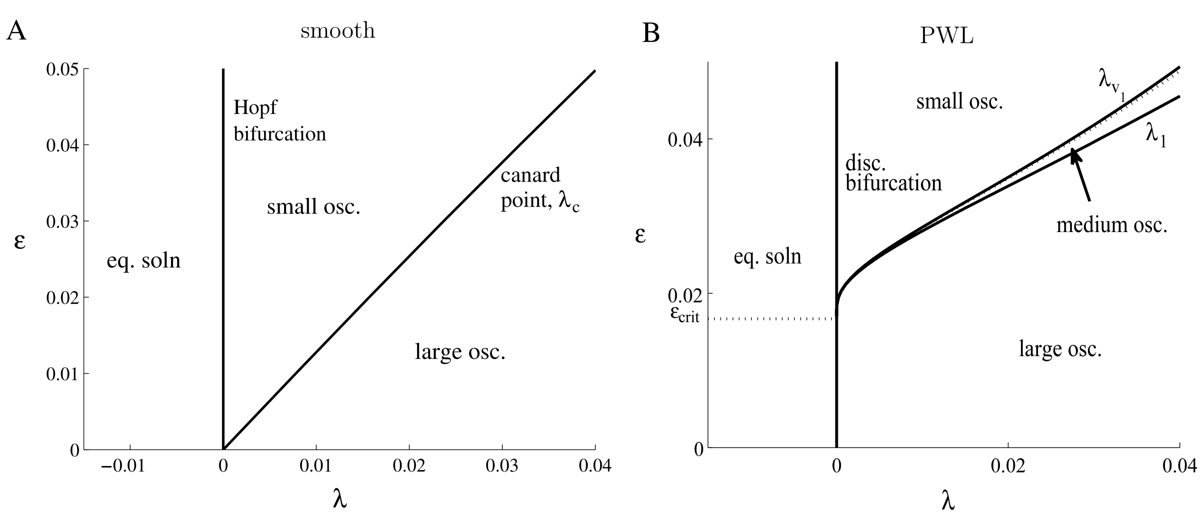

Fig. 3. Two parameter bifurcation diagrams of the smooth and PWL versions of (1) with the same parameter values as in Fig. 1. The smooth system has a well-defined canard point, λ_c [15,16], whereas for the PWL system we consider the two values, λ_{v_1} and λ_1, described in the text. In both panels we have indicated the attracting solution for each region bounded by the solid curves. The dotted curve in panel B corresponds to the approximation (14) derived below; ε_{crit} is given by (11). Note that in contrast to the remainder of this paper, in panel A the distinction between small and large oscillations is determined by λ_c and not (5).