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The synchronization transition in correlated oscillator populations

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ABSTRACT

The synchronization transition of correlated ensembles of coupled Kuramoto oscillators on sparse random networks is investigated. Extensive numerical simulations show that correlations between the native frequencies of adjacent oscillators on the network systematically shift the critical point as well as the critical exponents characterizing the transition. Negative correlations imply an onset of synchronization for smaller coupling, whereas positive correlations shift the critical coupling towards larger interaction strengths. For negatively correlated oscillators the transition still exhibits critical behaviour similar to that of the all-to-all coupled Kuramoto system, while positive correlations change the universality class of the transition depending on the correlation strength. Crucially, the paper demonstrates that the synchronization behaviour is not only determined by the coupling architecture, but also strongly influenced by the oscillator placement on the coupling network.

1. Introduction

The collective dynamics of synchronization has constituted a field of very active interest over the last couple of years. Typical areas where synchronization phenomena play an important role are manifold, ranging from fields as diverse as ecology, social dynamics, biological rhythms to fields like laser physics [1–3] and also power systems [4]. These real-world systems consist of many elementary units that interact. Since the discovery that the coupling networks are often non-trivial [5,6], the research into synchronization phenomena on complex networks has attracted significant attention [6,7].

So far most of the work in this field has dealt with understanding the influence of the topology of the interaction network on synchronization properties [6,7]. Studies of this problem have mainly concentrated on three approaches: (i) the master stability function approach [8] for achieving an understanding of the stability of the fully synchronized state for systems of identical chaotic oscillators; (ii) several analytical methods for studying the onset of synchronization [9,10]; and (iii) a numerical exploration of the properties of the synchronization transition of network ensembles [11–18]. These studies have provided much insight, demonstrating, e.g., that for symmetrical coupling small homogeneous uncliquish load-balanced networks facilitate the transition to complete synchronization [19]. In asymmetrically coupled systems, which have mainly been investigated through weighting schemes on undirected networks [20–23], homogeneous in-signals and balanced loads constitute two key indicators of an enhanced synchronizability.

In the second and third streams of this research the critical coupling that characterizes the onset of synchronization and its relation to structural properties of the coupling network have been explored, mainly using the Kuramoto model [2]

$$\dot{\phi}_i = \omega_i + \sigma \sum_j a_{ij} \sin(\phi_j - \phi_i) \tag{1}$$

as a well-understood model for use in the study of phase synchronization (see, e.g., [24] for a recent summary). In Eq. (1) the ϕ_i, $i = 1, \ldots, N$, describe the phases of N oscillators, the ω_i their native frequencies, σ the coupling strength and the matrix a_{ij} the topology of the coupling, i.e. the interaction network. Following, e.g., [9,15,25,26] we set $a_{ij} = 1$ if i and j are connected and $a_{ij} = 0$ otherwise. For a very good discussion of this choice of normalization see [17].

For all-to-all coupling $a_{i,j} = 1/(N-1)\ \forall i \neq j$ the model (1) exhibits a second-order phase transition from a desynchronized to a (partially) synchronized phase at some critical coupling strength $\sigma_c = 2/\pi g(0)$ [2], where $g(\cdot)$ is the distribution of the oscillator's native frequencies. Interestingly, the synchronization transition appears to be very similar to the mean-field type version of the model for some classes of complex networks such as the Strogatz–Watts small world model [11], random graphs, and even some kinds of scale-free networks [15]. However, for some kinds of degree distributions of scale-free networks the characteristics of the synchronization transition are found to depend on the degree heterogeneity [10].

Recently, as another approach, optimization techniques were used to investigate the relationship between correlations in oscillator placements, network architecture and synchronization [27–30]. The results from these studies suggest that a correlated oscillator placement can strongly affect a system's synchronization

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properties. More specifically, a grouping of oscillators on the network in such a way that the native frequencies of linked oscillators are anti-correlated has been shown to induce a transition to complete synchronization for low coupling, whereas a positively correlated oscillator arrangement requires more coupling strength for macroscopic synchronization to occur. These findings have recently been corroborated by a study of synchronization in bipolar population networks of Kuramoto oscillators [26].

The results in [27,30,26] suggest that the characteristics of the transition to synchronization may be different for positively or negatively correlated oscillator arrangements. However, this issue and the scaling with the system size have not been studied systematically so far and are the main subjects of the present study.

In our view, these results appear particularly relevant for evolved ecological or biological systems, where the synchronization of the individual elements might have been a determinant of the fitness of the system and thus have guided its evolution. In such a case, the evolution of the system will probably have been strongly affected by heterogeneity in the characteristics (i.e. native frequencies in the Kuramoto model) of the individual elements. Thus, the observed architecture of the system is not only characterized by the topology of the interaction network, but also strongly determined by the oscillator placement on the network. Similarly, as gains in the average degree of synchronization are at least as strongly marked by rearrangements in the oscillator placement as by the evolution of the network topology itself, as [27] highlights, introducing specific oscillator correlations may be easier to implement than a change in the overall network arrangement in technical applications where synchronization between the individual elements is desirable.

Correlations between the native frequencies of adjacent oscillators can be measured using a standard Pearson correlation coefficient

$$c_\omega = \frac{\sum_{i,j} a_{ij}(\omega_i - \langle\omega\rangle)(\omega_j - \langle\omega\rangle)}{\sum_{i,j}(\omega_i - \langle\omega\rangle)^2 a_{ij}}, \tag{2}$$

where $\langle\omega\rangle = 1/N \sum_i \omega_i$ is the average native frequency. It should be noted that strongly correlated oscillator placements are only possible on sparsely connected interaction networks. Generally, the more connected a network, the less correlated the oscillator placements that are possible, i.e. for a fully coupled system one trivially has $c_\omega = 0$. To the best of our knowledge, apart from in [27–29], correlations between network topology and oscillator placement and their influence on synchronization have not found any attention in the considerable literature on synchronization of complex networks; cf., e.g., [6,7] for a recent survey.

In this paper, we follow up on the above findings of our previous work and carry out a detailed analysis of the properties of the synchronization transition on random networks with tunable correlated oscillator placements. As will be shown below, correlations in the oscillator placement can affect both the critical point that separates the desynchronized from the synchronized phase, and the universality class of the synchronization transition. Results that elaborate on the dependence of the properties of the transition on the characteristics of the oscillator arrangement on the network are detailed below.

2. Analysis of the synchronization transition

To investigate the synchronization transition we follow a straightforward approach and construct random (undirected) Erdös–Rényi type graphs [31]. For technical reasons, to eliminate an irrelevant source of heterogeneity that otherwise would have to be averaged over, we choose the 'microcanonical' random graph model, where exactly $L = pN$ distinct randomly chosen pairs of nodes are connected by links [32]. Starting from an initially random oscillator placement we then carry out a simple optimization procedure to generate a correlated oscillator arrangement with a given correlation coefficient c^*_ω. This is achieved by randomly selecting pairs of nodes and considering swapping the associated native frequencies. Swaps are accepted if they lead to a correlation coefficient c_ω closer to the desired value c^*_ω, i.e. we minimize the difference $|c_\omega - c^*_\omega|$. The procedure is terminated if the correlation coefficient reaches the desired level of correlation within a small error tolerance, i.e. $|c_\omega - c^*_\omega| \ll \epsilon$, where we set $\epsilon = 10^{-4}$ for the rest of the study. Once the correlated oscillator ensemble is generated, we then integrate Eq. (1) numerically with initial conditions $\phi_i(t = 0)$ randomly selected from $(-\pi, \pi]$ and determine the standard order parameter

$$r(t)e^{i\psi} = \sum_j e^{i\phi_j(t)}. \tag{3}$$

The amplitude order parameter $r(t)$ is then averaged over time (after allowing for a sufficient time interval for relaxation) and then over several hundred different (correlated) oscillator frequencies ω_i and network configurations a_{ij}, where the native frequencies ω_i are selected uniformly at random from $[-1, 1]$ [33]. More precisely, we measure

$$r = \left\langle 1/T \sum_{t=T_{\mathrm{rel}}}^{T_{\mathrm{rel}}+T} r(t) \right\rangle, \tag{4}$$

where $\langle\cdot\rangle$ indicates the average over different initial conditions and the oscillator and network ensemble. As, e.g., in [11–13,16,18] a finite size scaling analysis is then carried out to determine the characteristics of the synchronization transition.

In the following analysis we focus on networks with average degree $\langle k\rangle = 3.5$, i.e. very sparsely connected networks which nevertheless already have a giant component that comprises more than 95% of all nodes. Clearly, oscillators cannot be placed on loops of odd length in a perfectly anti-correlated manner. Thus, large densities of short loops of odd length don't allow for strongly anti-correlated oscillator placements on the network. In this context, the sparse connectivity ensures an asymptotically vanishing number of triangles and in general low densities of short loops. It is worthwhile to point out that in order to highlight the effect of oscillator correlations on the synchronization transition in a regime in which the effect is strongest, all experiments have been carried out on very sparsely connected networks. In the light of this, comparisons to previous numerical and analytical work via mean-field approaches [12–14] are difficult.

Using the above optimization method, correlated oscillator placements with c_ω in the range between −0.6 and 0.6 can be generated. Even though the rest of the study is focused on only one value of the network connectivity $\langle k\rangle = 3.5$, we have also experimented with different link densities in this sparse regime. The results presented below are found to be prototypical and the qualitative statement appears to be generally valid for sparsely connected networks.

The precise determination of the transition between the desynchronized and the synchronized phase requires a careful consideration of finite size effects. Here we follow the approach of [11,16,18] with the finite size scaling ansatz

$$r(\sigma, N) = N^{-\alpha} F((\sigma - \sigma_c)N^{1/\nu}), \tag{5}$$

where the exponent $\alpha = \beta/\nu$ is related to the exponent β that describes the scaling of the order parameter r close to the critical point in the thermodynamic limit, i.e.

$$r \sim (\sigma - \sigma_c)^\beta. \tag{6}$$