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A correspondence between distances and embeddings for manifolds: New techniques for applications of the Abstract Boundary

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ABSTRACT

We present a one-to-one correspondence between equivalence classes of embeddings of a manifold (into a larger manifold of the same dimension) and equivalence classes of certain distances on the manifold. This correspondence allows us to use the Abstract Boundary to describe the structure of the 'edge' of our manifold without resorting to structures external to the manifold itself. This is particularly important in the study of singularities within General Relativity where singularities lie on this 'edge'. The ability to talk about the same objects, e.g., singularities, via different structures provides alternative routes for investigation which can be invaluable in the pursuit of physically motivated problems where certain types of information are unavailable or difficult to use.

1. Introduction

The study of singularities within General Relativity suffers from a unique problem in physics: there is no background metric in which the singularity exists. Yet our intuition wishes to describe these 'singularities' with a location and physical properties. There are also the additional problems of providing a coordinate independent definition of a singularity and a description of the full range of singular behaviour.

There are a number of boundary constructions which attempt to provide both a definition of and a location for singularities of space–times (see [1] for a review of the most notable boundary constructions, [2] for a review of the field in general and [3] for proofs of the most important applications). The three most common examples are the g-boundary [4], the b-boundary [5] and the c-boundary [6].[1] Each of these uses some aspects of the metric structure to identify 'missing' points from the space–time and then prescribes a method to re-attach them. Because each of these constructions uses the metric structure in this way, they each suffer from a variety of flaws. The Abstract Boundary (or a-boundary) [8] avoids these flaws by only using topological information in its construction.

The Abstract Boundary avoids using the metric structure by using embeddings as a kind of reference for the boundary structure. Specifically, it uses the set of all embeddings $\phi : \mathcal{M} \to \mathcal{M}_\phi$, where $\mathcal{M}$ is the manifold of our original space–time and $\mathcal{M}_\phi$ is a manifold of the same dimension as $\mathcal{M}$ (we shall refer to such an embedding as an envelopment), to construct a set of equivalence classes of boundary points of these embeddings. Each equivalence class corresponds to the representation

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[1] See the preprint, [7], for an up to date review of the c-boundary including work conducted at the same time as this paper.

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of a 'missing' point in a coordinate chart. Hence a 'missing' point may look very different depending on the chosen chart. Remarkably, a definition and location of a singularity can be retrieved from this rather general setup.

We note that there have been two important recent contributions in this area. The first is García-Parrado and Senovilla's isocausal boundary [9–11] which is, in part, inspired from the Abstract Boundary. The second is a number of recent developments of the c-boundary; we refer the reader to Sánchez' interesting paper [12]. In both cases their work was more directly concerned with causal structures.

The very nice thing about the a-boundary is that it avoids all the usual problems inherent in metrically constructed boundaries. Unfortunately, this comes at a cost. In particular, complete knowledge of the a-boundary is reliant on knowing all possible envelopments of $\mathcal{M}$. This reliance makes it almost impossible to construct the complete a-boundary for a general space–time. It should be emphasised, however, that this does not, in any way, hinder its utility in the investigation of problems related to singularities (e.g., [13]). In much the same way as one does not need to know all charts in the atlas of a manifold, so too one does not need the full Abstract Boundary to extract information about the 'edge' of space–times. This is more a matter of representation than of missing information.

This paper demonstrates that there is a one-to-one correspondence between the set of equivalence classes of envelopments and a set of equivalence classes of distances on the manifold $\mathcal{M}$. We give a short example of how this correspondence can be used to investigate the a-boundary structure of space–times. In a future paper the authors will demonstrate that this correspondence can be used to construct the complete a-boundary from this set of equivalence classes of distances. Thus the correspondence provides an alternative method for studying the a-boundary. While this does not solve the problem mentioned above, it does make it more readily accessible.

We also hope that this correspondence will be of interest to all mathematicians desiring to study the 'edges' of manifolds. The work below demonstrates that the a-boundary has a strong relationship to Cauchy structures, in the sense of [14], and thus also to the more normal boundaries, e.g., the Stone–Čech compactification, employed in topology.

Section 2 introduces the necessary background for the Abstract Boundary. Sections 3 and 4 present equivalence relations on the set of all envelopments of a manifold and a set of distances on a manifold, respectively, and discuss the relation of this work to Cauchy structures. The first relation describes when two envelopments provide the same information about the a-boundary. The second relation mirrors the ideas of the first, but on a set of distances rather than envelopments. Section 5 gives a one-to-one correspondence between the two sets of equivalence classes, thereby showing that what can be constructed using the first can also be constructed using the second. Section 6 presents a short example of how this correspondence can be used.

Our main result, contained in Section 5, is that the set of equivalence classes of envelopments, relevant for the Abstract Boundary, is in one-to-one correspondence with a set of equivalence classes of distances. So, in effect, the main result states that, in order to study the Abstract Boundary, one can use either envelopments or a certain subset of distances. The ability to employ distances when using the Abstract Boundary to investigate problems should provide both greater flexibility and accessibility.

To construct this correspondence we will use three homeomorphisms, between the closures, $\overline{\phi(\mathcal{M})}$, $\overline{\psi(\mathcal{M})}$, of the images of $\mathcal{M}$ under equivalent envelopments, ϕ, ψ, between the Cauchy completion, $\mathcal{M}^d$, $\mathcal{M}^{d'}$, of $\mathcal{M}$ of equivalent distances, d, d' and between the closure, $\overline{\phi(\mathcal{M})}$, of the image of $\mathcal{M}$ under an envelopment ϕ and the Cauchy completion of $\mathcal{M}$ with respect to a distance, d_ϕ, that is related to the envelopment. The existence of these homeomorphisms is ensured by Propositions 3.5, 4.8 and Corollary 4.9. Both the propositions show that certain functions have extensions into the completion of their domains. It is here that the theory of Cauchy spaces underlies our result as the functions we consider are not necessarily uniformly continuous and therefore their well known extension theorem does not apply. The needed generalisation of this extension theorem is expressed in the language of Cauchy spaces; see Sections 3.1 and 4.1.

2. Preliminary results and notation

We need a few results and definitions from previous papers about the Abstract Boundary; they are collected below for the convenience of the reader. We recommend that the reader refer to the cited papers for a detailed introduction to the subject.

Definition 2.1 (*See [8]*)**.** Let $\mathcal{M}$ and $\mathcal{M}'$ be manifolds of the same dimension. If there exists $\phi : \mathcal{M} \to \mathcal{M}'$ a C^∞ embedding, then $\mathcal{M}$ is said to be enveloped by $\mathcal{M}'$, $\mathcal{M}'$ is the enveloping manifold and ϕ is an envelopment. Since both manifolds have the same dimension, $\phi(\mathcal{M})$ is open in $\mathcal{M}'$.

Definition 2.2 (*See [8]*)**.** Let $\phi : \mathcal{M} \to \mathcal{M}_\phi$ be an envelopment. A non-empty subset B of $\partial(\phi(\mathcal{M}))$ is called a boundary set.

Definition 2.3 (*See [8]*)**.** A boundary set $B \subset \partial(\phi(\mathcal{M}))$ is said to cover another boundary set $B' \subset \partial(\psi(\mathcal{M}))$ if and only if for every open neighbourhood U of B in $\mathcal{M}_\phi$ there exists an open neighbourhood V of B' in $\mathcal{M}_\psi$ so that

$$\phi \circ \psi^{-1}(V \cap \psi(\mathcal{M})) \subset U.$$

We shall denote this partial order by $\triangleright$, so that B covers B' is written $B \triangleright B'$. If a boundary set B is such that it consists of a single point, i.e., $B = \{p\}$, then we shall simply write $p \triangleright B'$ rather than the more cumbersome $\{p\} \triangleright B'$.