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Phase description of nonlinear dissipative waves under space–time-dependent external forcing

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ABSTRACT

Based on the model system undergoing phase separation and chemical reactions, we investigate the dynamics of propagating dissipative waves under external forcing which is periodic both in space and time. A phase diagram for the entrained and non-entrained states under the external forcing is obtained numerically. Theoretical analysis in terms of phase description of the traveling waves is carried out to show that the transition between the entrained and the non-entrained states by changing the external frequency occurs either through a saddle–node bifurcation or through a Hopf bifurcation and that these two bifurcation lines are connected at a Bogdanov–Takens bifurcation point.

1. Introduction

Synchronization and entrainment of nonlinear oscillators under external periodic forcing have been studied for many years. It has been shown that the phase dynamics which introduces one phase variable for a limit cycle oscillation is very useful to understand those phenomena [1,2]. The time-evolution equation for the phase θ is given by

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \Omega - \omega + f(\theta), \tag{1}$$

where ω is the frequency of the limit cycle oscillation and Ω is that of the periodic external forcing. The function $f(\theta)$ is a 2π-periodic function. It is evident that Eq. (1) for $0 < \theta < 2\pi$ has a pair of time-independent solutions for small differences of $|\omega - \Omega|$. One is stable and the other is unstable. If the value $|\omega - \Omega|$ is increased by changing the external frequency Ω, the pair of solutions converges and disappears. This means that the bifurcation is a saddle–node bifurcation.

In comparison with these studies of nonlinear oscillators, nonlinear dissipative waves under external forcing have not been explored extensively despite the fact that existence of such waves are one of the most relevant self-organized phenomena far from equilibrium. Some recent studies towards this direction are given in Refs. [3–8]. In the previous papers [9,10], we addressed this problem not only for the external forcing but also for the feedback control. We carried out numerical simulations and theoretical analysis based on a model system in one dimension. In the present paper, we focus our analysis on the external forcing and investigate the entrained dynamics in further detail. In the next section (Section 2) we start with a description of our model system. Numerical results are shown in Section 3. The phase dynamics approach is given in Section 4. Section 5 is devoted to discussion.

2. Model equations

We start with the coupled set of equations for the local concentrations in a hypothetical ternary mixture where both phase separation and chemical reactions take place simultaneously. Let us assume that molecules A, B and C are adsorbed on a flat substrate with the local concentrations, ψ_A, ψ_B and ψ_C, respectively. The other chemical species involved in the chemical reactions are assumed to exist abundantly in the gas phase above the substrate, and the products are also assumed to dissolve quickly into the gas phase. Each lattice site of the substrate is occupied by one and only one molecule A, B or C. Any pair of molecules A and B that are nearest neighbors exchange their positions randomly with

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a certain probability, but C molecules do not participate in such exchanges. In this way, the condition $\psi_A + \psi_B + \psi_C = 1$ is satisfied in the continuum limit, while diffusion is exhibited by A and B molecules but not C molecules. When these molecules encounter other molecules in the gas phase, they undergo the chemical reactions $A \to B \to C \to A$ with the reaction rates γ_1, γ_2 and γ_3 respectively. It is assumed that the A and B species tend to segregate each other whereas the C component is neutral to both A and B. Then the time-evolution equations for the local concentrations $\psi = \psi_A - \psi_B$ and $\phi = \psi_A + \psi_B$ are given by [11]

$$\frac{\partial \psi}{\partial t} = \nabla^2[-\nabla^2\psi - \tau\psi + \psi^3] + a_1\psi + a_2\phi + a_3 + \Gamma(x,t), \quad (2)$$

$$\frac{\partial \phi}{\partial t} = b_1\psi + b_2\phi + b_3 + \Gamma(x,t). \quad (3)$$

The phase separation process is characterized by the parameter $\tau > 0$. The coefficients are given in terms of the reaction rates by

$$a_1 = -\left(\gamma_1 + \frac{\gamma_2}{2}\right), \quad (4)$$

$$a_2 = -\left(\gamma_1 - \frac{\gamma_2}{2} + \gamma_3\right), \quad (5)$$

$$a_3 = b_3 = \gamma_3, \quad (6)$$

$$b_1 = \frac{\gamma_2}{2}, \quad (7)$$

$$b_2 = -\left(\frac{\gamma_2}{2} + \gamma_3\right). \quad (8)$$

The function $\Gamma(x,t)$ stands for the external force which is moving steadily to the right

$$\Gamma(x,t) = \varepsilon\cos(q_f x - \Omega t), \quad (9)$$

with the strength ε, the wave number q_f and the frequency Ω [12]. Here we suppose that the system is exposed by illuminating light through a periodically arrayed slit and the slit moves at a constant velocity Ω/q_f. As a result, we assume that the reaction rate γ_3 is modified such that $\gamma_3 \to \gamma_3 + \Gamma$. In this way, the Γ term is added both in Eqs. (2) and (3) since $a_3 = b_3 = \gamma_3$ as Eq. (6). We have ignored a term $\Gamma\phi$ arising from the $\gamma_3\phi$ term in Eqs. (5) and (8) assuming a sufficiently small forcing ϵ.

We have studied earlier the solution of Eqs. (2) and (3) without the external forcing [11]. The uniform time-independent solution becomes unstable by increasing the parameter τ with fixing other parameters. Depending on the rate constants, e.g. γ_3, there are two possibilities. One is a Hopf bifurcation at a finite wave number. We have verified that a traveling wave appears above the threshold. The other is a Turing-type bifurcation beyond which a spatially periodic motionless pattern appears.

Throughout this paper, we will fix the parameters as $\tau = 1.6$, $\gamma_1 = 0.3$, $\gamma_2 = 0.16$ and $\gamma_3 = 0.05$. This set of the parameters are close to the Hopf bifurcation threshold $\tau = \tau_c = 1.46$ at a finite wave number $q = q_c \approx 0.9$ [11]. The frequency of oscillation at the bifurcation point is given by $\omega_c \approx 0.07$ and the external frequency Ω is varied around this critical frequency to investigate the dynamics under forcing.

3. Numerical simulations

We have carried out numerical simulations of Eqs. (2) and (3) with (9) in one dimension. The system size is $L = 20\pi$ with a periodic boundary condition and the space is divided into $N = 128$ cells with the cell size $\delta x = 20\pi/N$. This system size is almost commensurate with the critical spatial period of the traveling wave $\ell_c = 2\pi/q_c \approx 2\pi/0.9$. The wave number of the external force is fixed to be the same as q_c in order to avoid extra complications of dynamics. The explicit Euler scheme is employed with the time increment $\delta t = 0.001$. Initially we provide a wave propagating to the right without the external forcing and then, at a certain time instant, switch on the external force (9) which is also traveling to the right.

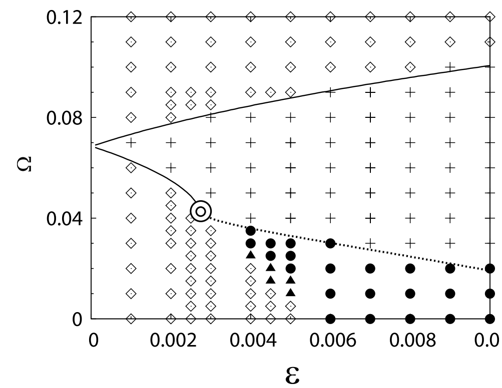

Fig. 1. Phase diagram for the entrainment with the external forcing traveling to the right on the $\varepsilon - \Omega$ plane. The meaning of the symbols is given in the text. The solid lines are the saddle–node bifurcation thresholds whereas the dotted line is the Hopf bifurcation threshold. These two lines are obtained from the phase equations of motion (12) and (13). The Bogdanov–Takens bifurcation point is indicated by the double circle.

Fig. 1 represents the phase diagram on the $\varepsilon - \Omega$ plane obtained numerically asymptotically in time. The traveling wave is completely entrained by the external force in the region filled by symbols (+) whereas it is not entrained in the region filled by $\diamond$. In the region indicated by $\bullet$, the wave trains undergo an oscillation trapped at the potential minima of the traveling external force. The space–time plot of these dynamics for $\varepsilon = 0.007$ is displayed in Fig. 2 where the gray scale indicates the magnitude of ψ. The entrained state ($\Omega = 0.07$) is shown in Fig. 2(a). Fig. 2(b) illustrates the drift state ($\Omega = 0.1$) where the wave speed is modulated periodically every time the external force catches up the traveling waves. Fig. 2(c) exhibits the trapped state ($\Omega = 0.02$) where each wave train moves back and forth propagating gradually to the right on an average. In the narrow region indicated by the black triangles in Fig. 1, propagation reversal occurs. That is, the wave propagating to the right starts to propagate to the left after applying the external force which is propagating to the right. The mechanism of this apparently strange phenomenon will be clarified in Section 4. In particular, see Fig. 6.

The above results are obtained in the situation that the external forcing is traveling to the same direction as the propagating wave. It should be noted, however, that Eqs. (2) and (3) without the external forcing have the waves traveling both to the right and to the left depending on the initial condition. Therefore, it is interesting to see what dynamics appears when the force moving to the opposite direction is applied. The phase diagram in such a case is obtained numerically as shown in Fig. 3 where the white circles indicates the region that the waves keep propagating to the initial direction with the periodic modulation by the external force traveling to the opposite direction. That is, the waves are not entrained. In the region indicated by other symbols, the waves change their propagating direction after switching on the external force and the asymptotic dynamics are the same as those in Fig. 1.

4. Phase equations

In order to clarify the dynamics in the phase diagram displayed in Fig. 1, we derive the phase equations of motion for the propagating waves under the external force. We represent the solutions of Eqs. (2) and (3) as

$$\psi = \psi_0 + \psi_1(t)\cos(q_c x - \Omega t + \theta_1(t)), \quad (10)$$

$$\phi = \phi_0 + \phi_1(t)\cos(q_c x - \Omega t + \theta_2(t)), \quad (11)$$