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Hamiltonian structures and their reciprocal transformations for the r-KdV–CH hierarchy

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ABSTRACT

The r-KdV–CH hierarchy is a generalization of the Korteweg–de Vries and Camassa–Holm hierarchies parameterized by $r + 1$ constants. In this paper we clarify some properties of its multi-Hamiltonian structures including the explicit expressions of the Hamiltonians, the formulae of the central invariants of the associated bihamiltonian structures and the relationship of these bihamiltonian structures with Frobenius manifolds. By introducing a class of generalized Hamiltonian structures, we present in a natural way the transformation formulae of the Hamiltonian structures of the hierarchy under certain reciprocal transformations, and prove the validity of the formulae at the level of dispersionless limit.

1. Introduction

In recent years progress has been made in the study of the problem of classification of bihamiltonian structures of a certain type; the associated bihamiltonian integrable hierarchies include in particular the well-known Korteweg–de Vries (KdV) hierarchy, the Camassa–Holm (CH) hierarchy, the Drinfeld–Sokolov hierarchies and so on [1–4]. For a given bihamiltonian structure defined on the formal loop space of an n-dimensional manifold M, a complete set of its invariants under the so-called Miura type transformations is obtained in [2,4]. It consists of a flat pencil of metrics defined on the manifold M and n functions of one variable; these functions are called the central invariants of the bihamiltonian structure. These invariants enable one to have a better understanding of the bihamiltonian structures and the associated integrable hierarchies. For most of the well-known bihamiltonian integrable hierarchies including the Drinfeld–Sokolov hierarchies associated to untwisted affine Lie algebras, the flat pencil of metrics is given by certain Frobenius manifold structures, and the central invariants are some constants. In particular, for the bihamiltonian structures of the Drinfeld–Sokolov hierarchies associated to the untwisted affine Lie algebras of A-D-E type, the central invariants are all equal to $\frac{1}{24}$ if one chooses the invariant bilinear form of the Lie algebra to be the normalized one [5,3]. This property is one of the most important characteristics of the integrable hierarchies that arise in 2D topological field theory and Gromov–Witten invariants [6,7,5,1,8–10].

On the other hand, to our knowledge the only known bihamiltonian integrable hierarchies with non-constant central invariants are special cases of the so-called r-KdV–CH hierarchy (see its definition given in the next section). This hierarchy is a generalization of the KdV hierarchy parameterized by an ordered set of $r + 1$ constants $\mathcal{P} = (a_0, a_1, \dots, a_r)$. Apart from

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the KdV hierarchy, it contains many other important integrable hierarchies as particular examples, such as the CH hierarchy, the AKNS hierarchy and the two-component Camassa–Holm (2-CH) hierarchy. In these cases the corresponding parameters are given by

$$\begin{aligned}
&\text{KdV} : r = 1, \mathcal{P} = (1, 0),\\
&\text{CH} : r = 1, \mathcal{P} = (0, 1),\\
&\text{AKNS} : r = 2, \mathcal{P} = (1, 0, 0),\\
&\text{2-CH} : r = 2, \mathcal{P} = (0, 0, 1).
\end{aligned}$$

The central invariants of the bihamiltonian structures of the KdV hierarchy and AKNS hierarchy are constant, while that of the CH hierarchy and the 2-CH hierarchy are not [11,12,4].

In some cases, bihamiltonian integrable hierarchies with different central invariants are related via certain type of transformations which change, unlike the Miura type transformations, also the independent variables. Such transformations are called reciprocal transformations, they are quite important in studying properties of solutions of the related integrable hierarchies (see [13–19] and references therein). A typical example is given by the relation between the KdV hierarchy and the CH hierarchy [15], the associated reciprocal transformation provides an efficient way to obtain exact solutions of the CH hierarchy by using the known solutions of the KdV hierarchy [15,16].

The main purpose of the present paper is to study, via the example of the r-KdV–CH hierarchy, the transformation rule of Hamiltonian structures under reciprocal transformations. A better understanding of such transformation rules is important in particular for the study of properties of the class of bihamiltonian integrable systems of Camassa–Holm type, and for the study of a generalized classification scheme for integrable hierarchies under reciprocal transformations.

The first step towards the generalization of the KdV hierarchy to the r-KdV–CH hierarchy was made by Martínez Alonso in [20], where he presented the r-KdV–CH hierarchy with the parameters $\mathcal{P} = (1, 0, \dots, 0)$ and proved its integrability by using the bihamiltonian structure of the hierarchy and the inverse scattering method. Antonowicz and Fordy studied in a series of important papers (see [21–23] and references therein) the spectral problem associated to the r-KdV–CH hierarchy with general parameters, they obtained $r + 1$ Hamiltonian operators associated to the spectral problem and pointed out that the compatibility of these Hamiltonian operators can be proved by using Fuchssteiner and Fokas' method of hereditary symmetry [24]. To fix the notations of the present paper, we first give in Section 2 the explicit formulation of the r-KdV–CH hierarchy with general parameters $\mathcal{P} = (a_0, \dots, a_r)$, and describe its $r + 1$ Hamiltonian structures including a formula of the Hamiltonians.

Properties of the bihamiltonian structures of the r-KdV–CH hierarchy were considered in [2], where a formula for the central invariants of these bihamiltonian structures was given without a proof. In Section 3 we fill the proof for the formula. We also specify in this section those bihamiltonian structures of the r-KdV–CH hierarchy which are associated to Frobenius manifolds. Recall that the bihamiltonian structures of the r-KdV–CH hierarchy admit hydrodynamic limits, and under certain conditions a bihamiltonian structures of hydrodynamic type is associated to the flat pencil of metrics defined on certain Frobenius manifold [7,25,1]. Here the notion of Frobenius manifold was invented by Dubrovin as a coordinate free formulation of the WDVV equation which arises in 2D topological field theory [26,27,6,7]. The rich geometry structures of Frobenius manifolds reveal in a natural way the close relationship between integrable hierarchies and 2D topological field theory.

Under reciprocal transformations an evolutionary PDE which possesses a local Hamiltonian structure will in general be transformed to a system with nonlocal Hamiltonian structures. For a Hamiltonian system of hydrodynamic type, such transformation properties were studied by Ferapontov and Pavlov in [14] where the nonlocal Hamiltonian structure was obtained from the expressions of the transformed systems. In order to generalize their results to Hamiltonian structures with dispersive terms such as the ones for the r-KdV–CH hierarchy, we introduce in Section 4 a class of generalized Hamiltonian structures which includes in particular the type of nonlocal Hamiltonian structures that we are interested in, and we also give some important properties of such generalized Hamiltonian structures while leaving their proofs to a separate publication [28]. We then apply these results in Section 5 to the study of properties of the Hamiltonian structures of the r-KdV–CH hierarchy under certain reciprocal transformations.

We give some concluding remarks in the last section.

2. The *r*-KdV–CH hierarchy

In this section we recall the definition of the r-KdV–CH hierarchy and their Hamiltonian structures.

Let M be a contractible manifold with local coordinates $w^0, \dots, w^{r-1}$, and φ be a smooth map from the circle $S^1 = \mathbb{R}/\mathbb{Z}$ to M

$$\varphi : S^1 \to M, \quad x \mapsto (w^0(x), \dots, w^{r-1}(x)).$$

We denote the derivatives $\partial_x w^i(x), \partial_x^2 w^i(x), \dots$ by $w^i_x, w^i_{xx}, \dots$, and denote $\partial_x^k w^i(x) = w^{i,k}$ in general. Let $\bar{\mathcal{A}}$ be the polynomial ring

$$\bar{\mathcal{A}} = C^\infty(M)[\epsilon\, w^i_x, \epsilon^2\, w^i_{xx}, \dots].$$