Journal of Geometry and Physics 59 (2009) 1244–1257

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Integrable structure of melting crystal model with two $q$-parameters

Kanehisa Takasaki *

*Graduate School of Human and Environmental Studies, Kyoto University Yoshida, Sakyo, Kyoto, 606-8501, Japan*

ARTICLE INFO

*Article history:*
Received 15 March 2009
Received in revised form 28 May 2009
Accepted 8 June 2009
Available online 11 June 2009

*MSC:*
35Q58
81R12
82B99

*Keywords:*
Plane partition
Free fermion
Quantum torus
Toda hierarchy
$q$-difference analogue

ABSTRACT

This paper explores integrable structures of a generalized melting crystal model that has two $q$-parameters $q_1, q_2$. This model, like the ordinary one with a single $q$-parameter, is formulated as a model of random plane partitions (or, equivalently, random 3D Young diagrams). The Boltzmann weight contains an infinite number of external potentials that depend on the shape of the diagonal slice of plane partitions. The partition function is thereby a function of an infinite number of coupling constants $t_1, t_2, \ldots$ and an extra one $Q$. There is a compact expression of this partition function in the language of a 2D complex free fermion system, from which one can see the presence of a quantum torus algebra behind this model. The partition function turns out to be a tau function (times a simple factor) of two integrable structures simultaneously. The first integrable structure is the bigraded Toda hierarchy, which determines the dependence on $t_1, t_2, \ldots$. This integrable structure emerges when the $q$-parameters $q_1, q_2$ take special values. The second integrable structure is a $q$-difference analogue of the 1D Toda equation. The partition function satisfies this $q$-difference equation with respect to $Q$. Unlike the bigraded Toda hierarchy, this integrable structure exists for any values of $q_1, q_2$.



## 1. Introduction

The *melting crystal model* is a model of statistical physics and describes a melting corner of a crystal that fills the first quadrant of the 3D Euclidean space. The complement of the crystal in the first quadrant may be thought of as a 3D analogue of Young diagrams. These 3D Young diagrams can be represented by *plane partitions*. Thus the melting crystal model can be formulated as a model of *random plane partitions*.

This model has been applied to string theory [1] and gauge theory [2,3]. From the point of view of gauge theory, the partition function of the melting crystal model is a 5D analogue of the instanton sum of 4D $\mathcal{N} = 2$ supersymmetric Yang–Mills theory [4–6]. (Curiously, the 4D instanton sum also resembles a generating function the Gromov–Witten invariants of the Riemann sphere [7,8].) This analogy will need further explanation, because the 4D instanton sum is a sum over ordinary partitions rather than plane partitions. The fact is that one can use the idea of *diagonal slicing* [9] to rewrite the partition function of the melting crystal model to a sum over ordinary partitions [2]. Comparing these two models of *random partitions*, one can consider the melting crystal model as a kind of *q-deformation* of the 4D instanton sum. Here $q$ is a parameter of the melting crystal model related to temperature.

In our previous work [10] (see also the review [11]), we introduced a set of external potentials into this model, and identified an integrable structure that lies behind this partition function. Namely, the partition function, as a function of the coupling constants $t_1, t_2, \ldots$ of potentials, turns out to be equal to a tau function (times a simple factor) of the Toda hierarchy [12,13]. Moreover, the tau function satisfies a set of constraints that reduces the full Toda hierarchy to the so-called *1D Toda hierarchy*. Though a similar fact was known for the 4D instanton sum [14–16], we found that the partition function of the melting crystal model can be treated in a more direct manner. We derived these results on the basis of a

* Tel.: +81 757536738; fax: +81 757536694.
*E-mail address:* takasaki@math.h.kyoto-u.ac.jp.

0393-0440/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2009.06.006

fermionic formula of the partition function [14]. A technical clue is a set of algebraic relations among the basis of a quantum torus (or cylinder) algebra realized by fermions. These relations enabled us to rewrite the partition function to a tau function of the Toda hierarchy.

In the present paper, we generalize these results to a melting crystal model with two $q$-parameters $q_1, q_2$ [17]. Actually, since the potentials have another $q$-parameter $q$, this model has altogether three $q$-parameters $q_1, q_2$ and $q$; letting $q_1 = q_2 = q$, we can recover the previous model.

Our goal is two-fold. Firstly, we elucidate an integrable structure that emerges when $q_1$ and $q_2$ satisfy the relations $q_1 = q^{1/N_1}$ and $q_2 = q^{1/N_2}$ for a pair of positive integers $N_1$ and $N_2$. The partition function in this case turns out to be, up to a simple factor, a tau function of (a variant of) the *bigraded Toda hierarchy* of type $(N_1, N_2)$ [18], which is also a reduction of the Toda hierarchy. Secondly, without such condition on the parameters $q_1$, $q_1$ and $q$, we show that the partition function satisfies a $q$-difference analogue [19–22] of the Toda equation with respect to yet another coupling constant $Q$. In the gauge theoretical interpretation, $Q$ is related to the energy scale $\Lambda$ of supersymmetric Yang–Mills theory.

This paper is organized as follows. Section 2 is a review of combinatorial aspects of the usual melting crystal model. The model with two $q$-parameters is introduced in the end of this section. Section 3 is an overview of the fermionic formula of the partition function. After reviewing these basic facts, we present our results on integrable structures in Sections 4 and 5. Section 4 deals with the bigraded Toda hierarchy, and Section 5 the $q$-difference Toda equation. We conclude this paper with Section 6.

## 2. Melting crystal model

In the following, we shall use a number of notions and results on partitions, Young diagrams and Schur functions. For details of those combinatorial tools, we refer the reader to Macdonald's book [23]. See also Bressoud's book [24] for related issues and historical backgrounds.

### 2.1. Simplest model

Let us start with a review of the ordinary melting crystal model with a single parameter $q$ ($0 < q < 1$). As a model of statistical physics, this system can take various states with some probabilities, and these states are represented by plane partitions.

Plane partitions are 2D analogues of ordinary (1D) partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$, and denoted by 2D arrays

$$\pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

of nonnegative integers $\pi_{ij}$ (called *parts*) such that only a finite number of parts are non-zero and the inequalities

$$\pi_{ij} \geq \pi_{i,j+1}, \quad \pi_{ij} \geq \pi_{i+1,j}$$

are satisfied. Let $|\pi|$ denote the sum

$$|\pi| = \sum_{i,j=0}^{\infty} \pi_{ij}$$

of all parts $\pi_{ij}$.

Such a plane partition $\pi$ represents a 3D Young diagram in the first quadrant $x, y, z \geq 0$ of the $(x, y, z)$ space. In this geometric interpretation, $\pi_{ij}$ is equal to the height of the stack of cubes over the $(i, j)$-th position of the base $(x, y)$ plane. Therefore $|\pi|$ is equal to the volume of the 3D Young diagram.

In the formulation of the melting crystal model, the complement of the 3D Young diagram in the first quadrant embodies the shape of a partially melted crystal. We assume that such a crystal has energy proportional to $|\pi|$. Consequently, the partition function of this system is given by the sum

$$Z = \sum_{\pi} q^{|\pi|}$$

of the Boltzmann weight $q^{|\pi|}$ over all plane partitions $\pi$.

### 2.2. Diagonal slicing

We can convert this model of random plane partitions to a model of random partitions by diagonal slicing. This idea originates in the work of Okounkov and Reshetikhin [9] on a model of stochastic process (Schur process).

Let $\pi(m)$ ($m \in \boldsymbol{Z}$) denotes the partition that represents the Young diagram obtained by slicing the 3D Young diagram along the diagonal plane $x - y = m$ in the $(x, y, z)$ space. In terms of the parts $\pi_{ij}$ of the plane partition, these diagonal slices can be defined as

$$\pi(m) = \begin{cases} (\pi_{i,i+m})_{i=1}^{\infty} & \text{if } m \geq 0 \\ (\pi_{j-m,j})_{j=1}^{\infty} & \text{if } m < 0. \end{cases}$$