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Probing a subcritical instability with an amplitude expansion: An exploration of how far one can get

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ABSTRACT

We explore methods to locate subcritical branches of spatially periodic solutions in pattern forming systems with a nonlinear finite-wavelength instability. We do so by means of a direct expansion in the amplitude of the linearly least stable mode about the appropriate reference state which one considers. This is motivated by the observation that for some equations fully nonlinear chaotic dynamics has been found to be organized around periodic solutions that do not simply bifurcate from the basic (laminar) state. We apply the method to two model equations, a subcritical generalization of the Swift–Hohenberg equation and a novel extension of the Kuramoto–Sivashinsky equation that we introduce to illustrate the abovementioned scenario in which weakly chaotic subcritical dynamics is organized around periodic states that bifurcate "from infinity" and that can nevertheless be probed perturbatively. We explore the reliability and robustness of such an expansion, with a particular focus on the use of these methods for determining the existence and approximate properties of finite-amplitude stationary solutions. Such methods obviously are to be used with caution: the expansions are often only asymptotic approximations, and if they converge their radius of convergence may be small. Nevertheless, expansions to higher order in the amplitude can be a useful tool to obtain qualitatively reliable results.

1. Introduction

Many non-equilibrium systems show spatio-temporal instabilities of some kind: ripples on sand, convection rolls in fluids, turbulence in pipe flows, patterns in crystal growth, etc. If they are driven far enough away from equilibrium (usually quantified by some control parameter), a homogeneous initial state (say, a flat bed of sand, a laminar flow or a straight front) becomes unstable with respect to spatial perturbations of a certain wavelength. Often, perturbations with a wavenumber around a critical wavenumber start to grow, and the system ends up in an inhomogeneous state. This state may feature regular stationary or oscillatory patterns, travelling waves, or even spatiotemporal chaos or turbulence. Such *finite-wavelength instabilities* and the patterns they give rise to have been the focus of much research in the past few decades [1–8].

As is well known, there are a number of ways in which the transition from a homogeneous state to a patterned state can occur. Three of the most important ones are depicted schematically in Fig. 1. In (a) we sketch the common *supercritical* transition scenario, in which a stable pattern branch bifurcates off the homogeneous state at the point at which the homogeneous steady state becomes linearly unstable at some critical value of the control parameter. This scenario occurs frequently when the nonlinearities in the system lead to saturation. The amplitude of the pattern vanishes as the control parameter approaches its critical value from above. Close to the transition point, the amplitude generally scales as the square root of the distance to the transition point. A well-known example of this type of transition is the transition to rolls in Rayleigh–Bénard convection [1]. Fig. 1(b) depicts the case of a *subcritical* bifurcation: the system becomes linearly unstable beyond a critical value of the control parameter, but even below this point, there are nontrivial finite amplitude pattern solutions. The amplitude no longer vanishes when the critical point is approached from above. This type of behavior is found for example in Rayleigh–Bénard convection with non-Boussinesq effects [9,5], and in in many other systems without an "up–down" symmetry, like 2-dimensional reaction-diffusion systems with a Turing instability [10,5]. In Fig. 1(c) we finally sketch the case which is sometimes referred to as a *bifurcation from infinity* [11]: the homogeneous state is linearly stable for *all* values of the control parameter, but for sufficiently large control parameters there exists a branch of finite amplitude nontrivial solutions which in

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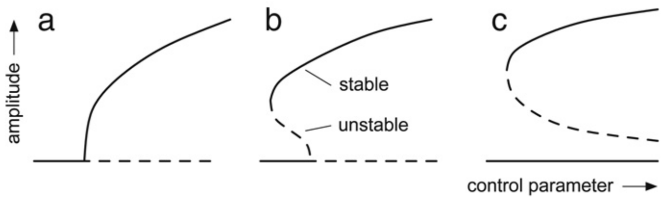

Fig. 1. Typical bifurcation diagrams for (a) a supercritical transition, (b) a subcritical transition and (c) a subcritical bifurcation from infinity. Solid lines denote linearly stable states, dashed lines are linearly unstable. Note that in practice the nonlinear solutions on the upper branch are sometimes not actually stable, but in such cases they can still organize the dynamics. This happens for instance in the transition to turbulence in Couette and pipe flow [12,13] and in the equation we construct and analyse in Section 4.

practice govern the dynamics of the system under some conditions. The lower (unstable) branch defines a kind of threshold amplitude: if perturbations are smaller than this amplitude, the system returns to the homogeneous state; if they are larger, the system ends up on the upper branch. The best known example of this scenario is the transition to turbulence in Newtonian fluids in plane Couette or Poiseuille pipe flow, although the nature of the "stable" and "unstable" branches is not at all clear in this case [12,13]. Two of us recently proposed that the same scenario may apply to shear flows of non-Newtonian viscoelastic fluids [14,15]. Note that while in the figure we indicate the upper branch to consist of stable solutions, this is often not the case in practice. One example is given by the exact two-dimensional nonlinear states in the form of travelling waves that were found in Newtonian plane channel flow [16]. On the upper branch, these solutions are stable in two dimensions, but they are unstable when an infinitesimally small three-dimensional perturbation is introduced [17]. Another example is Newtonian turbulence in pipe flows where three-dimensional nonlinear solutions play an important role in the dynamics, even though they are themselves unstable [12,13].

The partial differential equations that usually describe pattern-forming systems cannot be solved analytically in general. However, as the bifurcation diagrams already suggest, near a transition it is often possible to find a reduced description of the spatially periodic or travelling-wave solutions in terms of just the amplitude of the pattern. For supercritical transitions, the *amplitude equation* approach has been very successful [1–8].

For strongly subcritical transitions, however, this approach essentially breaks down. At the transition, the stable branch already has a nonzero amplitude, and the usual expansion in principle does not work, at least not for the most relevant stable (upper) branch. The lower branch of unstable solutions still grows as in the supercritical case, so unstable states can be found perturbatively sufficiently close to the transition (and thus, threshold amplitudes of perturbations). If the subcritical character is sufficiently weak, it is often possible to adapt the expansion to find also the stable solutions, as for example in Rayleigh–Bénard convection with non-Boussinesq effects [9]. The expansion is then formally no longer consistent, but works in practice because effectively there is a another small parameter (e.g., the smallness of the non-Boussinesq effects).

From a more formal point of view, one might argue that one simply should not use amplitude expansions to probe subcritical bifurcations and especially the bifurcations from infinity of Fig. 1(c) that motivate us, since the amplitude expansion, *which is only an asymptotic expansion*, can clearly not be trusted to give reliable results about the existence and stability of finite-amplitude patterns. In practice, however, such a strict point of view is not the most constructive one. After all, if one is investigating a new problem about which not much is known a priori, one does not necessarily know in advance whether patterns one observes are due to some supercritical or subcritical transition, or even a bifurcation from infinity—one actually does a calculation to find out what the nature of the problem is! Suppose then one finds in an amplitude expansion that the sign of the cubic nonlinearity signals that there is no saturation of the pattern amplitude at the lowest nontrivial level, in other words, that the bifurcation is not supercritical. Should one then simply stop at that point because the amplitude expansion formally cannot handle such a situation?

Clearly, such a defeatist attitude is not to be expected from an applied researcher who is eager to understand the nonlinear behavior of the problem at hand. In practice, if there is reason to believe on physical grounds that the transition is weakly subcritical, even though there may be no a priori small parameter in the equations that suggests this, such a practitioner of nonlinear science may want to try, nevertheless, to calculate the next (fifth order) term in the expansion, in the hope of being able to estimate how weak the subcritical character really is, and how large the amplitudes of the nonlinear pattern actually might be. And if such a calculation is done, one faces the question as to how reliable this estimate actually is and what the optimal truncation (if any) of the expansion might be.

Two of us recently faced a similar dilemma in a study of the nonlinear stability of viscoelastic shear flows [14,15,18] where we suspected, on physical grounds, the relevance of the bifurcation from infinity scenario of Fig. 1(c). Motivated by the expectation that the lower (unstable) branch – which determines the nonlinear instability threshold – would actually be close to the horizontal axis for intermediate values of the control parameter, and that the smallness of the transition amplitude could play the role of an intrinsic small parameter hidden in the problem, an amplitude expansion up to eleventh order was performed to estimate the nonlinear threshold. In this case, it actually does appear that useful information can be extracted from analyzing the behavior of the expansion to such high orders.

This paper is motivated by these observations and by our own experience, that even though hard and generally valid statements are difficult to make about the behavior of intrinsically asymptotic expansions, it is very profitable to get a better feel for how far one may push amplitude expansions to probe such intrinsically subcritical transitions. One of the main aims of this paper is to explore the possible signatures for failure or success of this pragmatic approach. Indeed, in studying these issues, we have empirically found that it is sometimes possible to push the expansions further by focusing on a limited question, like the existence and nature of a nonlinear (subcritical) branch of solutions. We discuss this method and its basis, and compare it to the results one obtains from a more straightforward amplitude expansion. Moreover, in order to illustrate that such expansion methods can even be useful in cases in which the nonlinear solutions that one can probe perturbatively are unstable but nevertheless important for the dynamics, we introduce a new simple model based on two coupled equations whose bifurcation diagram corresponds to the "bifurcation from infinity" case of Fig. 1(c), and which mimics a case in which Kuramoto–Sivashinsky-like chaos [19–21] is organized around exact periodic solutions. The gross features of the turbulent nonlinear branch of this equation are indeed captured well with our amplitude expansion, a finding that gives hope for our earlier work on nonlinear visco-elastic instabilities [15].

We stress here at the outset that our goal is rather limited. First of all, the equations we study are only used as exploratory examples. Secondly, we *neither* aim *nor* claim to investigate the full nonlinear dynamics of these equations; instead, we will focus simply on determining the presence and location of the subcritical branches of periodic solutions in the approximation so that only the amplitude of one mode is retained. The stability of the branches is hence only studied within this subspace of periodic solutions