



# Some connectedness problems in positively curved Finsler manifolds

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## ABSTRACT

This paper studies some connectedness problems under the positivity hypothesis of various curvatures ( $k$ -Ricci and flag curvature). Our approach uses Morse Theory for general end conditions (see [Ioan Radu Peter, The Morse index theorem where the ends are submanifolds in Finsler geometry, *Houston J. Math.* 32 (4) (2006) 995–1009]). Some previous results related to the flag curvature were obtained in [Ioan Radu Peter, A connectedness principle in positively curved Finsler manifolds, in: H. Shimada, S. Sabau (Eds.), *Advanced Studies in Pure Mathematics, Finsler Geometry, Sapporo 2005–In Memory of Makoto Matsumoto*, Mathematical Society of Japan, 2007]. Some results from Riemannian geometry are extended to the Finsler category also. The Finsler setting is much more complicated and the difference between Finsler and Riemann settings will be emphasized during the paper.

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## 1. Introduction

Connectedness principles are well known in algebraic and Riemannian geometry. Recently such kinds of principles have been developed in the Riemann or Kähler manifolds (see [7,8]). In a previous paper the first author generalizes to a Finsler setting connectedness principles in the case of positive flag curvature, see [18]. In this paper the case of positive  $k$ -Ricci curvature is considered. Namely we develop a connectedness principle in the case of positive  $k$  Ricci curvature for embedded submanifolds with a large asymptotic index.

In the Finslerian category the situation is much more complicated than in the Riemannian context. The variation of the energy applied to a geodesic with the ends on submanifolds naturally gives rise to a second fundamental form (see [17]). A submanifold is totally geodesic (that is geodesics of the submanifold are also geodesics for the ambient manifold) is equivalent to the statement that the second fundamental form vanishes holds only for Berwald spaces, because the reference vector of the second fundamental form (which appears in the connection coefficients) is not tangent to the submanifold.

We define the asymptotic index using the second fundamental form and the results are proved using the asymptotic index. But the results concerning totally geodesic submanifolds are true for Berwald spaces (in these spaces the asymptotic index is equal to the dimension of submanifold iff the submanifold is totally geodesic).

In the Riemannian case results of this type are obtained in terms of the asymptotic index, totally geodesic submanifolds or the extrinsic curvature of a submanifold (see [4,7,9] for the last concept). In this paper, using Morse theory in the Finslerian, the results involving asymptotic index are proved in the Finsler spaces, the results concerning totally geodesic submanifolds are proved in the Berwald–Finsler category and the results involving the extrinsic curvature are not treated because even in Berwald spaces, where the reference vector is irrelevant for the connection coefficients and also for the curvature tensor, the inner products which appear in the flag curvature have a dependence on the reference vector.

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In recent years the global behaviour of Ricci and flag curvature was extensively studied by Bao and Robles [3], Rademacher [19], and Shen [20].

## 2. Main results

Now we state the main connectedness theorems of the paper. Their proofs will follow after [Theorems 5 and 7](#), Section 5.

**Theorem 1.** Let  $M$  be an  $m$ -dimensional Finsler manifold of positive  $k$ -th Ricci curvature, and let  $f = (f_1, f_2) : N_1 \times N_2 \rightarrow M \times M$ , with  $f_j : N_j \rightarrow M$  an isometric immersion of a compact manifold with asymptotic index  $\nu_{f_j}, j = 1, 2$ . Then the following properties are true (we denote  $\nu_f = \nu_{f_1} + \nu_{f_2}$ ):

- (1) If  $\nu_f > m + k - 1$ , then  $f^{-1}(\Delta) \neq \emptyset$ .
- (2) If  $\nu_f > m + k$  and  $M$  is simply connected, then  $f^{-1}(\Delta)$  is connected.
- (3) For  $i < \nu_f - m - k + 1$  there is an exact sequence

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1f)_* - (p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)).$$

- (4) For  $i < \nu_f - m - k + 1$ , there are natural isomorphisms

$$\pi_i(N_1, f^{-1}(\Delta)) \rightarrow \pi_i(M, N_1)$$

for  $i \leq \nu_f - m$  and a surjection for  $i = \nu_f - m + 1$ . Here  $\pi_i(N_j, f^{-1}(\Delta))$  is understood as the  $i$ -th homotopy group of the composition map  $f^{-1}(\Delta) \hookrightarrow N \xrightarrow{p_j} N_j$ .

In the case when  $f$  is a pair of immersions we have:

**Theorem 2.** Let  $M$  be an  $m$ -dimensional Finsler manifold of positive  $k$ -th Ricci curvature, and let  $f = (f_1, f_2) : N_1 \times N_2 \rightarrow M \times M$ , with  $f_j : N_j \rightarrow M$  an isometric immersion of a compact manifold with asymptotic index  $\nu_{f_j}, j = 1, 2$ . Then the following properties are true (we denote  $\nu_f = \nu_{f_1} + \nu_{f_2}$ ):

- (1) If  $\nu_f \geq m + k - 1$ , then  $f^{-1}(\Delta) \neq \emptyset$ .
- (2) If  $\nu_f \geq m + k$  and  $M$  is simply connected, then  $f^{-1}(\Delta)$  is connected. If  $f = (f_1, f_1)$  where  $f_1$  is an embedding, then
- (3) For  $\nu_f \geq m + k + i - 1$  there is an exact sequence

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1f)_* - (p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)).$$

- (4) For  $i \leq \nu_f - m - k + 1$ , there are natural isomorphisms

$$\pi_i(N_1, f^{-1}(\Delta)) \rightarrow \pi_i(M, N_1)$$

for  $i \leq \nu_f - m$  and a surjection for  $i = \nu_f - m + 1$ . Here  $\pi_i(N_j, f^{-1}(\Delta))$  is understood as the  $i$ -th homotopy group of the composition map  $f^{-1}(\Delta) \hookrightarrow N \xrightarrow{p_j} N_j$ .

For positive flag curvature from [Theorems 5 and 8](#) we have

**Theorem 3** ([18]). Let  $M$  be an  $m$ -dimensional compact Finsler manifold of positive flag curvature and  $\Delta$  the diagonal of  $M \times M$ . Consider an isometric immersion  $f : N \rightarrow M \times M$  of a closed manifold with asymptotic index  $\nu_f$ . The following statements hold:

- (1) If  $\nu_f > m$ , then  $f^{-1}(\Delta) \neq \emptyset$ .
- (2) If  $\nu_f > m + 1$  and  $M$  is simply connected, then  $f^{-1}(\Delta)$  is connected.
- (3) For  $\nu_f > m + i$  the following sequence of homotopy groups

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1f)_* - (p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta)) \longrightarrow \dots$$

is exact.

In the case where  $f$  is not a correspondence but a pair of immersions we have the following stronger result:

**Theorem 4** ([18]). Under the assumptions of [Theorem 3](#) if in addition  $N = N_1 \times N_2$  and  $f = (f_1, f_2)$  with asymptotic index  $\nu_f$ , then

- (1) If  $\nu_f \geq m$  then  $f^{-1}(\Delta) \neq \emptyset$ .
- (2) If  $\nu_f \geq m + 1$  and  $M$  is simply connected, then it follows that  $f^{-1}(\Delta)$  is connected. If  $f = (f_1, f_1)$  with  $f_1$  embedding then
- (3) For  $\nu_f \geq m + i$  the following sequence of homotopy groups

$$\pi_i(f^{-1}(\Delta)) \longrightarrow \pi_1(N) \xrightarrow{(p_1f)_* - (p_2f)_*} \pi_i(M) \longrightarrow \pi_{i-1}(f^{-1}(\Delta))$$

is exact.

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