Journal of Geometry and Physics 58 (2008) 996–1006

Contents lists available at ScienceDirect

Journal of Geometry and Physics

journal homepage: www.elsevier.com/locate/jgp

# Gauge invariance, charge conservation, and variational principles

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ARTICLE INFO

*Article history:*
Received 20 October 2007
Received in revised form 10 February 2008
Accepted 6 March 2008
Available online 14 March 2008

*JGP SC:*
Classical field theory
Global analysis
Analysis on manifolds

*MSC:*
58E30
58J70
70S10
70S15

*Keywords:*
Gauge theories
Symmetries
Conservation laws
Variational principles

ABSTRACT

We present new results on the correspondence between symmetries, conservation laws and variational principles for field equations in general non-abelian gauge theories. Our main result states that second order field equations possessing translational and gauge symmetries and the corresponding conservation laws are always derivable from a variational principle. We also show by the way of examples that the above result fails in general for third order field equations.



## 1. Introduction and main results

The classical Noether's theorem establishes essentially a one-to-one correspondence between the symmetries and conservation laws of a system of partial differential equations admitting a variational principle. In 1977, Takens [28] considered and rigorously formulated the following novel aspect of Noether's theorem: Let $\mathfrak{g}$ be a Lie algebra of vector fields defined on the space of independent and dependent variables, and suppose that a system of differential equations is invariant under $\mathfrak{g}$ and that each element in $\mathfrak{g}$ generates a conservation law for the system. Does it then follow that the system arises from a variational principle, i.e., that it is the Euler–Lagrange expression of some Lagrangian function? In his original paper Takens considered the question for second order scalar equations, systems of linear equations and metric field theories. Subsequently, Takens' results on second order scalar equations and on systems of linear equations were substantially generalized by Anderson and Pohjanpelto [3,4,26]. We refer to [3] in particular for more background material and motivation on Takens' problem.

Apart from the papers listed above, the literature dealing with the existence of variational principles for systems of differential equations admitting a Lie algebra of symmetries and the corresponding conservation laws is mainly limited to

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doi:10.1016/j.geomphys.2008.03.006

classical field theories, where the symmetry group is the infinite dimensional group of coordinate transformations of the underlying manifold and the conservation laws express the vanishing of the covariant divergence (or some variant of it) of the field equations. The classification results of Cartan [7], Vermeil [29], and Weyl [30] imply that second order quasi-linear field equations for the metric tensor possessing the symmetries and conservation laws of the Einstein equations necessarily arise from a variational principle. This result was later generalized to general second order equations for the metric tensor and to third order equations in the 3-dimensional case by Lovelock, [19,21], whose results were subsequently extended to metric-scalar [13], metric-vector [20] and metric-bivector [22] theories. More recently, pure vector field theories with the symmetry group consisting of spatial translations and $U(1)$ gauge transformations were treated in [5].

In this paper we investigate the relationship between symmetries, conservation laws and variational principles for gauge theories with a general structure group on the $n$-dimensional Euclidean space $\mathbb{R}^n$. Gauge theories with a non-Abelian structure group play a central role in quantum field theories by providing a unified framework for the description of electromagnetism and the weak and strong forces [10], and in geometry as the pivotal ingredient in the Donaldson and Seiberg-Witten theories [8,23]. Recently, higher order gauge theories, i.e., theories with field equations or order $k \geq 3$, have been developed in classical [9,17] as well as in quantum settings [11], [12, p. 217].

The primary goal of the present paper is to identify conditions under which a system of gauge field equations, admitting translational and gauge symmetries and the associated conservation laws, arises from a variational principle. In contrast to the results of Cartan, Vermeil, Weyl and Lovelock, field equations of gauge theories, even in low orders, are not specified by these conditions, which rules out solving Takens' question in this situation by a brute force classification process. However, as is well known, the vanishing of the classical Helmholtz conditions for a system of differential equations guarantees the existence of a Lagrangian for the system, and our problem is rendered tractable by an analysis of these conditions for systems with the prescribed symmetries and conservation laws.

Write $(x^i)$ for the coordinates on $\mathbb{R}^n$ and let $\{e_\alpha\}$ be a basis of the Lie algebra $\mathfrak{g}$ of an $r$-dimensional Lie group $G$ with structure constants $c^\alpha_{\beta\gamma}$. A gauge field

$$A = A^\alpha_a(x^i)\mathrm{d}x^a \otimes e_\alpha$$

is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^n$, where the $A^\alpha_a$ stand for the components of $A$. The gauge field $A$ is subject to a system of $k$-th order partial differential equations

$$T^a_\alpha = T^a_\alpha(x^i, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_k}) = 0, \quad a = 1, \ldots, n, \alpha = 1, \ldots, r,$$

where $A^\beta_{b,j_1j_2\cdots j_l}$ denotes the derivative of $A^\beta_b$ with respect to the independent variables $x^{j_1}, x^{j_2}, \ldots, x^{j_l}$. The operator $T^a_\alpha$ is locally variational if it can be written as the Euler–Lagrange expression

$$T^a_\alpha = E^a_\alpha(L) = \frac{\partial L}{\partial A^\alpha_a} - D_{i_1}\frac{\partial L}{\partial A^\alpha_{a,i_1}} + D_{i_1}D_{i_2}\frac{\partial L}{\partial A^\alpha_{a,i_1i_2}} - \cdots$$

of some locally defined Lagrangian $L = L(x^j, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_l})$, where $D_i$ denotes the standard coordinate total derivative operator.

In this paper we consider the following classes of symmetries and conservation laws for a differential operator $T^a_\alpha$.

*Symmetries*

[S1] The operator $T^a_\alpha$ is invariant under the infinitesimal group of spatial translations

$$\mathfrak{t}(n) = \left\{ \mathbf{t} = a^i \frac{\partial}{\partial x^i} \,\middle|\, (a^i) \in \mathbb{R}^n \right\}. \tag{1}$$

[S2] The operator $T^a_\alpha$ is invariant under the infinite dimensional group of infinitesimal gauge transformations

$$\mathfrak{ga}(n) = \left\{ Q_\varphi = \left(\varphi^\alpha_{,a} + c^\alpha_{\beta\gamma}A^\beta_a\varphi^\gamma\right) \frac{\partial}{\partial A^\alpha_a} \,\middle|\, \varphi \in C^\infty(\mathbb{R}^n, \mathfrak{g}) \right\}. \tag{2}$$

*Conservation laws*

[C1] There are functions $t^i_p = t^i_p(x^j, A^\beta_b, A^\beta_{b,j_1}, A^\beta_{b,j_1j_2}, \ldots, A^\beta_{b,j_1j_2\cdots j_l})$ such that, for each $p = 1, 2, \ldots, n$,

$$A^\alpha_{a,p}T^a_\alpha = D_i(t^i_p).$$

[C2] The covariant divergence of the operator $T^a_\alpha$ vanishes identically,

$$\nabla_a T^a_\alpha = D_a T^a_\alpha + c^\gamma_{\alpha\beta}A^\beta_a T^a_\gamma = 0.$$

Our main result is the following.

**Theorem 1.** *Suppose that the differential operator $T^a_\alpha$ has symmetries* [S1], [S2] *and conservation laws* [C1]. *Then $T^a_\alpha$ is locally variational if*