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Multiplication of solutions for linear overdetermined systems of partial differential equations

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ABSTRACT

A large family of linear, usually overdetermined, systems of partial differential equations that admit a multiplication of solutions, i.e, a bi-linear and commutative mapping on the solution space, is studied. This family of PDE's contains the Cauchy–Riemann equations and the cofactor pair systems, included as special cases. The multiplication provides a method for generating, in a pure algebraic way, large classes of non-trivial solutions that can be constructed by forming convergent power series of trivial solutions.

1. Introduction

In this paper we study a wide class of linear first order systems of partial differential equations, that allow a bi-linear multiplication in the space of solutions. The simplest example is the Cauchy–Riemann equations. We know that two holomorphic functions, $f = V + \mathrm{i}\tilde{V}$ and $g = W + \mathrm{i}\tilde{W}$, can be multiplied in order to produce a new holomorphic function $fg = VW - \tilde{V}\tilde{W} + \mathrm{i}(V\tilde{W} + \tilde{V}W)$. In terms of the Cauchy–Riemann equations

$$\begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{\partial \tilde{V}}{\partial y} \\ \dfrac{\partial V}{\partial y} = -\dfrac{\partial \tilde{V}}{\partial x} \end{cases}$$

this multiplication can be expressed in the following way: two solutions $(V, \tilde{V})$ and $(W, \tilde{W})$ prescribe, in a bi-linear way, a new solution $(VW - \tilde{V}\tilde{W}, V\tilde{W} + \tilde{V}W)$. From the basic theory of holomorphic functions, we know that any solution of the Cauchy–Riemann equations can be expressed locally as a convergent power series of a simple solution with respect to the described multiplication.

The Cauchy–Riemann equations provide the simplest example of a system of PDE's that has a multiplication on its solution set, but there are more sophisticated examples. One such example is the multiplication of cofactor pair systems, discovered by Lundmark in [5].

A cofactor pair system (or bi-cofactor system) is a dynamical system $\ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h, h = 1, \dots, n$, on a (pseudo-) Riemannian manifold, such that the force F has two different cofactor formulations $F = (\operatorname{cof} J)^{-1}\nabla V = (\operatorname{cof} \tilde{J})^{-1}\nabla\tilde{V}$,

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where J and $\tilde{J}$ are independent special conformal Killing tensors of type $(1, 1)$, V and $\tilde{V}$ are smooth real-valued functions, $\operatorname{cof} J = (\det J)J^{-1}$, and ∇ is the gradient $((\nabla V)^i = g^{ij}\partial_j V)$. Cofactor pair systems have several desirable properties, in general they are completely integrable, they admit a bi-Hamiltonian formulation, and they are equivalent (or correspondent) to separable Lagrangian systems [1,2,6–9].

A cofactor pair system is characterized by a pair of functions V and $\tilde{V}$, and a pair of special conformal Killing tensors J and $\tilde{J}$, that satisfy the relation

$$(\operatorname{cof} J)^{-1}\nabla V = (\operatorname{cof} \tilde{J})^{-1}\nabla\tilde{V}. \tag{1}$$

For fixed special conformal Killing tensors J and $\tilde{J}$, the Eq. (1) constitutes a system of first order linear PDE's for two functions V and $\tilde{V}$. In [5], Lundmark found that Eq. (1) allows a multiplication of solutions. When $n = 2$ the multiplication formula is given by

$$(V, \tilde{V}) * (W, \tilde{W}) = \left(VW - \det(\tilde{J}^{-1}J)\tilde{V}\tilde{W},\, V\tilde{W} + \tilde{V}W - \operatorname{tr}(\tilde{J}^{-1}J)\tilde{V}\tilde{W}\right),$$

where $(V, \tilde{V})$ and $(W, \tilde{W})$ are solutions of (1). We see that when $\det(\tilde{J}^{-1}J)$ and $\operatorname{tr}(\tilde{J}^{-1}J)$ are not both constant, we can choose trivial (constant) solutions $(V, \tilde{V})$ and $(W, \tilde{W})$ of (1) and obtain non-trivial solutions through the multiplication. When $n > 2$ a multiplication also exists, but one has to consider the related parameter-dependent system

$$\left(\operatorname{cof}\ (J + \mu\tilde{J})\right)^{-1}\nabla V_\mu = (\operatorname{cof} \tilde{J})^{-1}\nabla\tilde{V},$$

which can also be written as

$$(\tilde{J}^{-1}J + \mu I)\nabla V_\mu = \det(\tilde{J}^{-1}J + \mu I)\nabla\tilde{V},$$

where V_μ is polynomial in the real parameter μ (note that throughout this paper, we use the notation V_μ rather than $V(\mu)$ to indicate dependence on the parameter μ).

The most interesting property of this multiplication is that it provides a tool for producing new cofactor pair systems from known ones. Especially, infinite families of separable potentials can be constructed. For example, the Jacobi, Neumann, and parabolic families of separable potentials are all constructed in [10] through a recursive process that is a special case of the multiplication of cofactor pair systems.

Remark 1. In [3], equations of the form (1), considered on a real or complex vector space where J and $\tilde{J}$ are constant matrices, are studied, and the general analytic solution is described.

In order to gain better understanding of this multiplication, systems of the form

$$(X + \mu I)\nabla V_\mu = \det(X + \mu I)\nabla\tilde{V}, \tag{2}$$

defined on a general (pseudo-) Riemannian manifold, were studied in [4], without referring to any underlying dynamical system. By analyzing the corresponding equations at each degree of μ in Eq. (2), it becomes obvious that the equation is satisfied if and only if the degree of V_μ is n and the left hand side $(X + \mu I)\nabla V_\mu$ can be written as a product of the scalar $\det(X + \mu I)$ and some 1-form which is constant in μ. We can therefore rewrite Eq. (2) as

$$(X + \mu I)\nabla V_\mu \equiv 0 \pmod{\det(X + \mu I)}. \tag{3}$$

It turned out that the system (2) allows for a multiplication of solutions, similar to the one existing for cofactor pair systems, if and only if the tensor X satisfies the equation

$$(X + \mu I)\nabla\det(X + \mu I) = \det(X + \mu I)\nabla\operatorname{tr}(X + \mu I). \tag{4}$$

Several classes of solutions of (4) where discovered, and it became apparent that systems of the form (1) and the Cauchy–Riemann equations only constitute special cases of a much larger family of systems of PDE's that admit a multiplicative structure on the solution space.

It was also remarked in [4], that by considering more general systems than (2), one finds other new classes of systems that allow multiplication of solutions. In this paper, we will examine that subject. The linear systems of PDE's that we consider are in general impossible to solve explicitly, but the multiplication provides a non-trivial superposition principle (on top of the ordinary linear superposition) that, for any two solutions, prescribes a new solution in a bi-linear and pure algebraic way. With this superposition principle, large classes of new solutions can be generated from known solutions. In particular, we can construct non-trivial solutions by forming convergent power series of a simple solution. The question then arises for which systems of linear PDE's these power series constitute all solutions, like in the case of the Cauchy–Riemann equations where all holomorphic functions admit a power series representation. Besides providing us with more systems of PDE's that admit a multiplicative structure on the solution set, the generalization helps us to better understand the multiplication for the systems already known (in particular the puzzling multiplication of cofactor pair systems).

This paper is organized as follows. In Section 2 we formulate an abstract framework for characterizing the class of systems of PDE's that admit multiplication. We define the $*$-operator and give a characterization of those systems that admit $*$-multiplication on the set of solutions. The multiplication provides a method for generating, in a pure algebraic and non-trivial way, new solutions from known solutions. A second formulation of the systems, using related matrices, is introduced.