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New infinite-dimensional multiple-symmetry groups for the Einstein–Maxwell-dilaton–axion theory

Ya-Jun Gao *

Department of Physics, Bohai University, Jinzhou 121013, Liaoning, People's Republic of China

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ABSTRACT

The symmetry structures of stationary axisymmetric Einstein–Maxwell-Dilaton–Axion (EMDA) theory are further studied. By using the so-called extended double (ED)-complex function method, the usual Riemann–Hilbert (RH) problem is extended to an ED-complex formulation. Two pairs of ED RH transformations are constructed and they are verified to give infinite-dimensional multiple-symmetry groups of the EMDA theory; each of these symmetry groups has the structure of a semidirect product of a Kac–Moody group $\widehat{Sp(4,R)}$ and a Virasoro group. Moreover, the infinitesimal forms of these RH transformations are calculated and they are found to give exactly the same results as previous work; this demonstrates that the two pairs of ED RH transformations in this paper provide exponentiations of all the infinitesimal symmetries in our previous paper. The finite forms of symmetry transformations given in the present paper are more important and useful for theoretical studies and new solution generation, etc.

1. Introduction

Recently, much attention has been attracted to the studies of symmetries for the dimensionally reduced low energy effective (super)string theories (e.g. [1–24]) owing to their importance in theoretical and mathematical physics. Such effective string theories describe various interacting matter fields coupled to gravity; the Einstein–Maxwell-dilaton–axion (EMDA) theory (see e.g. [3,11,12,14,17,18,23,24]) is a typical and important model of this kind. Some analogies between the EMDA theory and the reduced vacuum Einstein theory have been noted. However, the mathematical structures of the EMDA theory are much more complicated. For example, many scalar functions in pure gravity correspond, formally, to matrix ones in the effective string theory; thus the non-commuting property of the matrices gives rise to essential complications for the further study of the latter. Moreover, some important and useful formulas in some studies of the reduced vacuum gravity (e.g. [25–29]) will have no general analogues in the EMDA theory, so deeper researches and further extended studying methods are needed.

The present paper is a continuation of our previous papers [23,24]. In [23,24], we found the doubleness symmetry of the stationary axisymmetric (SAS) EMDA theory. Further, by using the so-called extended double (ED)-complex function

* Tel.: +86 416 3400148; fax: +86 416 3400149.
E-mail address: yajgao@bhu.edu.cn.

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method [29], we constructed the ED-complex 4×4 matrix H, F-potentials and established two pairs of ED-complex Hauser–Ernst (HE)-type linear systems. On the basis of these linear systems, we explicitly constructed new infinitesimal multiple-symmetry transformations for the SAS EMDA theory and verified that they constitute quadruple infinite-dim ('-dim' stands for '-dimensional' or ' dimensions' here and hereafter) Lie algebras, each of which is a semidirect product of the Kac–Moody $\widehat{sp(4, R)}$ and Virasoro algebras. However, for theoretical studies and new solution generation, etc., it is more important and useful to find finite symmetry transformations of the theory considered; this is the main aim of the present paper.

In Section 2, some related concepts and notation for the ED-complex functions [29], the ED-complex H, F-potentials and two pairs of HE-type linear systems for EMDA theory [24] are briefly recalled. In Section 3, we construct a pair of ED Riemann–Hilbert (RH) transformations relating to the first pair of HE-type linear systems and then prove that they are indeed double-symmetry transformations of the EMDA theory. In Section 4, the equivalent integral equation formulations are given and the infinite-dim group structures of the ED RH transformations are verified. In Section 5, we find that a double-duality mapping can be introduced and a second pair of ED RH transformations relating to the second pair of HE-type linear systems can be given. In Section 6, infinitesimal forms of the given ED RH transformations are calculated, which give exactly the same results as our previous paper [24]. These demonstrate that the two pairs of ED RH transformations in the present paper provide exponentiations of all the infinitesimal symmetry transformations given in [24]. Finally, Section 7 gives a summary and discussion.

2. ED-complex H, F-potentials and HE-type linear systems for EMDA theory

For later use, here we briefly recall some related concepts and notation for the ED-complex function [29] as well as the ED-complex H, F-potentials and HE-type linear systems for EMDA theory [24].

2.1. ED-complex function [29]

Let i and J denote, respectively, the ordinary and the ED imaginary unit. We shall concern ourselves mainly with some special values of J, i.e. $J = j$ ($j^2 = -1,\ j \neq \pm i$) or $J = \varepsilon$ ($\varepsilon^2 = +1,\ \varepsilon \neq \pm 1$). If a series $\sum_{n=0}^{\infty} |a_n|,\ a_n \in \mathbf{C}$ (ordinary complex number), is convergent, then $a(J) = \sum_{n=0}^{\infty} a_n J^{2n}$ is called an ED ordinary complex number, which can correspond to a pair $(a_C,\ a_H)$ of ordinary complex numbers, where $a_C := a(J = j)$, $a_H := a(J = \varepsilon)$. When $a(J)$ and $b(J)$ are both ED ordinary complex numbers, $c(J) = a(J) + Jb(J)$ is called an ED-complex number; it can correspond to a pair $(c_C,\ c_H)$, where $c_C := c(J = j) = a_C + jb_C$, $c_H := c(J = \varepsilon) = a_H + \varepsilon b_H$. We define $a(J) := \mathrm{Re}_{\mathrm{ED}}(c(J))$, $b(J) := \mathrm{Im}_{\mathrm{ED}}(c(J))$. If $a(J)$ and $b(J)$ are real, we call them double-real and call the corresponding $c(J)$ simply a double-complex number.

We would like to point out that, from the above definitions, J should be taken as an indeterminate rather than a discrete variable. The ED-complex method can be regarded as some "deformation" theory, in which J plays the role of a "deformation parameter" (or analytical link; cf. [30] for the non-extended case). By doubleness symmetry we in fact mean the symmetry property of the considered theory under this "deformation". We call it an ED-complex method only because in most of its applications (e.g. in the present paper) we are mainly interested in the cases of $J = j$ and $J = \varepsilon$.

All ED-complex numbers with usual addition and multiplication constitute a commutative ring. Corresponding to the two imaginary units J and i in this ring, we have two complex conjugations: ED-complex conjugation "$\star$" and ordinary complex conjugation "$-$"

$$c(J)^{\star} := a(J) - Jb(J), \qquad \overline{c(J)} := \overline{a(J)} + J\overline{b(J)}. \tag{2.1}$$

These imply that $J^{\star} = -J$, $\bar{J} = J$, $i^{\star} = i$, $\bar{i} = -i$. If $a(J)$ and $b(J)$ are ED ordinary complex functions of some ordinary complex variables $z_1, \ldots, z_n$, then $c(z_1, \ldots, z_n;\ J) = a(z_1, \ldots, z_n;\ J) + Jb(z_1, \ldots, z_n;\ J)$ is called an ED-complex function. We describe $c(z_1, \ldots, z_n; J)$ as continuous, analytical, etc. if $a(z_1, \ldots, z_n;\ J)$ and $b(z_1, \ldots, z_n;\ J)$ both, as ordinary complex functions, have the same properties. For an ED-complex matrix $W(J)$, we define

$$W(J)^{+} := [W(J)^{\star}]^{\top}, \tag{2.2}$$

where "$\top$" denotes transposition. The ED imaginary unit commutation operator "$\circ$" is defined by

$$\circ : J \longrightarrow \overset{\circ}{J}, \qquad \overset{\circ}{j} = \epsilon, \qquad \overset{\circ}{\epsilon} = j. \tag{2.3}$$

Obviously, $\overset{\circ}{J}$ is the ED imaginary unit, too.

2.2. ED-complex H, F-potentials and HE-type linear systems [24]

The EMDA action, which describes the bosonic sector of the heterotic string in 4-dim, and contains a metric $g_{\mu\nu}$ (signature $+---$, $\mu, \nu = 0, 1, 2, 3$), a $U(1)$ vector field A_{μ}, a Kalb–Ramond antisymmetric tensor field $B_{\mu\nu}$ and a dilaton field ϕ, can be written as [11,12]