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Review

Variational equations on mixed Riemannian–Lorentzian metrics

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A class of elliptic–hyperbolic equations is placed in the context of a geometric variational theory, in which the change of type is viewed as a change in the character of an underlying metric. A fundamental example of a metric which changes in this way is the extended projective disc, which is Riemannian at ordinary points, Lorentzian at ideal points, and singular on the absolute. Harmonic fields on such a metric can be interpreted as the hodograph image of extremal surfaces in Minkowski 3-space. This suggests an approach to generalized Plateau problems in three-dimensional space-time via Hodge theory on the extended projective disc. Analogous variational problems arise on Riemannian–Lorentzian flow metrics in fiber bundles (twisted nonlinear Hodge equations), and on certain Riemannian–Lorentzian manifolds which occur in relativity and quantum cosmology. The examples surveyed come with natural gauge theories and Hodge dualities. This paper is mainly a review, but some technical extensions are proven.

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1. Introduction: The projective disc

In a small circle of paper, you shall see as it were an epitome of the whole world.
Giambattista Della Porta, 1589, on the *camera obscura*

Analysis on Beltrami's projective disc model for hyperbolic space is in one sense very old mathematics. Beltrami introduced the projective disc in 1868 as one of the earliest Euclidean models for non-Euclidean space [9]; see also [10, 113]. But it also arises in the context of some new mathematics related to variational problems in Minkowski space and Hodge theory on pseudo-Riemannian manifolds. In practice Beltrami's construction amounts to equipping the unit disc centered at the origin of coordinates in $\mathbb{R}^2$ with the distance function

$$ds^2 = \frac{(1-y^2)\,dx^2 + 2xy dx dy + (1-x^2)\,dy^2}{(1-x^2-y^2)^2}. \tag{1}$$

Integrating ds along geodesic lines in polar coordinates, we find that the distance from any point in the interior of the unit disc to the boundary of the disc is infinite, so the unit circle becomes the *absolute*: the curve at projective infinity; *cf.* [49], Section 9.1.

It is natural to ask how to interpret the so-called *ideal* points in the complement of the unit disc in $\mathbb{R}^2$. It has been known for a long time that such points are not merely allowed by the projective disc model in a formal sense, but are actually useful in classical geometric constructions. For example, lines tangent to the unit disc can be used to characterize orthogonal lines within the disc, and certain families of translated lines inside the disc attain their simplest representation as a rotation about an ideal point. Although these classical geometric operations on ideal points are well known [112,64], geometric analysis on domains which include ideal points is still very incompletely understood.

In this review we use Beltrami's model as a point of reference from which to survey aspects of geometric variational theory on *mixed Riemannian–Lorentzian* domains, on which the signature of the metric changes sign along a smooth hypersurface. (The metric underlying (1) changes from Riemannian to Lorentzian on the Euclidean unit circle.) In Section 2 we consider variational problems which reduce to the existence of harmonic fields on the extended projected disc — that is, solutions to the *Hodge equations*

$$d\alpha = \delta\alpha = 0 \tag{2}$$

on a domain of $\mathbb{R}^2$ which includes the closed unit disc as a proper subset and is equipped with the Beltrami metric (1). Here d is the exterior derivative, with formal adjoint δ, acting on a 1-form α [83]. The crucial technical problem for the equations of Section 2 is the existence of solutions to boundary-value problems; this question is addressed in Section 3. Sections 4 and 5 are concerned with the related topics of duality and gauge invariance. These topics motivate the study of gauge-invariant, potentially elliptic–hyperbolic systems in fiber bundles. In Section 6 we briefly review similar systems that arise in relativity and cosmology, and the relation of those systems to other elliptic–hyperbolic equations on manifolds which have been studied by mathematicians.

On the extended projective disc the Hodge equations are no longer uniformly elliptic, but are elliptic on ordinary points inside the unit disc, hyperbolic on ideal points in the $\mathbb{R}^2$-complement of the disc, and parabolic on the unit circle, which is a singularity of the manifold. One might wonder why anyone would want to study such a peculiar system. There are at least two motivations for doing so:

(i) to learn what the geometry of the Beltrami disc reveals about elliptic–hyperbolic partial differential equations, and
(ii) to learn what elliptic–hyperbolic partial differential equations on the Beltrami disc reveal about the geometry of space-time.

Regarding the first motivation, there is no canonical way to decide what constitutes a natural boundary-value problem for an equation that changes from elliptic to hyperbolic type on a smooth curve. Historically, physical analogies have been the main tool, chiefly analogies to the physics of compressible flow [12]. However, it is also possible to approach the problem using a geometric analogy, which we will briefly describe.

The highest-order terms of any linear second-order partial differential equation on a domain $\Omega \subset \mathbb{R}^2$ can be written in the form

$$Lu = \alpha\,(x,y)\,u_{xx} + 2\beta(x,y)u_{xy} + \gamma(x,y)u_{yy}, \tag{3}$$

where (x, y) are coordinates on Ω; α, β, and γ are given functions; $u = u\,(x, y)$. Traditionally, the question of whether the equation is elliptic, hyperbolic, or parabolic has been identified with the question of whether the discriminant

$$\Delta\,(x,y) = \alpha\gamma - \beta^2 \tag{4}$$

is respectively positive, negative, or zero. If the discriminant is positive on part of Ω and negative elsewhere on Ω, then the equation associated with the operator L is said to be of *mixed elliptic–hyperbolic type*. The curve on which the equation