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Unique continuation results for Ricci curvature and applications

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Abstract

Unique continuation results are proved for metrics with prescribed Ricci curvature in the setting of bounded metrics on compact manifolds with boundary, and in the setting of complete conformally compact metrics on such manifolds. Related to this issue, an isometry extension property is proved: continuous groups of isometries at conformal infinity extend into the bulk of any complete conformally compact Einstein metric. Relations of this property with the invariance of the Gauss–Codazzi constraint equations under deformations are also discussed.

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1. Introduction

In this paper, we study certain issues related to the boundary behavior of metrics with prescribed Ricci curvature. Let M be a compact $(n+1)$-dimensional manifold with compact non-empty boundary ∂M. We consider two possible classes of Riemannian metrics g on M. First, g may extend smoothly to a Riemannian metric on the closure $\bar{M} = M \cup \partial M$, thus inducing a Riemannian metric $\gamma = g|_{\partial M}$ on ∂M. Second, g may be a complete metric on M, so that ∂M is "at infinity". In this case, we assume that g is conformally compact, i.e. there exists a defining function ρ for ∂M in M such that the conformally equivalent metric

$$\widetilde{g} = \rho^2 g \tag{1.1}$$

extends at least C^2 to ∂M. The defining function ρ is unique only up to multiplication by positive functions; hence only the conformal class $[\gamma]$ of the associated boundary metric $\gamma = \bar{g}|_{\partial M}$ is determined by (M, g).

The issue of boundary regularity of Riemannian metrics g with controlled Ricci curvature has been addressed recently in several papers. Thus, [4] proves boundary regularity for bounded metrics g on M with controlled Ricci curvature, assuming control on the boundary metric γ and the mean curvature of ∂M in M. In [16], boundary regularity

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is proved for conformally compact Einstein metrics with smooth conformal infinity; this was previously proved by different methods in dimension 4 in [3], cf. also [5].

One purpose of this paper is to prove a unique continuation property at the boundary ∂M for bounded metrics or for conformally compact metrics. We first state a version of the result for Einstein metrics on bounded domains.

Theorem 1.1. *Let (M, g) be a $C^{3,\alpha}$ metric on a compact manifold with boundary M, with induced metric $\gamma = g|_{\partial M}$, and let A be the 2nd fundamental form of ∂M in M. Suppose the Ricci curvature Ric_g satisfies*

$$\mathrm{Ric}_g = \lambda g, \tag{1.2}$$

where λ is a fixed constant.

Then (M, g) is uniquely determined up to local isometry and inclusion, by the Cauchy data (γ, A) on an arbitrary open set U of ∂M.

Thus, if (M_1, g_1) and (M_2, g_2) are a pair of Einstein metrics as above, whose Cauchy data (γ, A) agree on an open set U common to both ∂M_1 and ∂M_2, then after passing to suitable covering spaces $\bar{M}_i$, either there exist isometric embeddings $\bar{M}_1 \subset \bar{M}_2$ or $\bar{M}_2 \subset \bar{M}_1$ or there exists an Einstein metric $(\bar{M}_3, g_3)$ and isometric embeddings $(\bar{M}_i, g_i) \subset (\bar{M}_3, g_3)$. Similar results hold for metrics which satisfy other covariant equations involving the metric to second order, for example the Einstein equations coupled to other fields; see Proposition 3.7.

For conformally compact metrics, the second fundamental form A of the compactified metric $\bar{g}$ in (1.1) is umbilic, and completely determined by the defining function ρ. In fact, for conformally compact Einstein metrics, the higher order Lie derivatives $\mathcal{L}_N^{(k)}\bar{g}$ at ∂M, where N is the unit vector in the direction $\bar{\nabla}\rho$, are determined by the conformal infinity $[\gamma]$ and ρ up to order $k < n$. Supposing ρ is a geodesic defining function, so that $\|\bar{\nabla}\rho\| = 1$, let

$$g_{(n)} = \frac{1}{n!}\mathcal{L}_N^{(n)}\bar{g}. \tag{1.3}$$

More precisely, $g_{(n)}$ is the nth term in the Fefferman–Graham expansion of the metric g; this is given by (1.3) when n is odd, and in a similar way when n is even, cf. [18] and Section 4. The term $g_{(n)}$ is the natural analog of A for conformally compact Einstein metrics.

Theorem 1.2. *Let g be a C^2 conformally compact Einstein metric on a compact manifold M with C^∞ smooth conformal infinity $[\gamma]$, normalized so that*

$$\mathrm{Ric}_g = -ng. \tag{1.4}$$

Then the Cauchy data $(\gamma, g_{(n)})$ restricted to any open set U of ∂M uniquely determine (M, g) up to local isometry and determine $(\gamma, g_{(n)})$ globally on ∂M.

The recent boundary regularity result of Chruściel et al. [16], implies that (M, g) is C^∞ polyhomogeneous conformally compact, so that the hypotheses of Theorem 1.2 imply the term $g_{(n)}$ is well defined on ∂M. A more general version of Theorem 1.2, without the smoothness assumption on $[\gamma]$, is proved in Section 4, cf. Theorem 4.1. For conformally compact metrics coupled to other fields, see Remark 4.5.

Of course neither Theorem 1.1 or 1.2 hold when just the boundary metric γ on $U \subset \partial M$ is fixed. For example, in the context of Theorem 1.2, by [20,16], given any C^∞ smooth boundary metric γ sufficiently close to the round metric on S^n, there is a smooth (in the polyhomogeneous sense) conformally compact Einstein metric on the $(n+1)$-ball B^{n+1}, close to the Poincaré metric. Hence, the behavior of γ in U is independent of its behavior on the complement of U in ∂M.

Theorems 1.1 and 1.2 have been phrased in the context of "global" Einstein metrics, defined on compact manifolds with compact boundary. However, the proofs are local, and these results hold for metrics defined on an open manifold with boundary. From this perspective, the data (γ, A) or $(\gamma, g_{(n)})$ on U determine whether Einstein metric g has a global extension to an Einstein metric on a compact manifold with boundary (or conformally compact Einstein metric), and how smooth that extension is at the global boundary.

A second purpose of the paper is to prove the following isometry extension result which is at least conceptually closely related to Theorem 1.2. However, while Theorem 1.2 is valid locally, this result depends crucially on global properties.