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Turbiner's conjecture in three dimensions

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Abstract

We prove a modified version of Turbiner's conjecture in three dimensions and we give a counter-example to the original conjecture. The Lie algebraic Schrödinger operators corresponding to flat metrics of a certain restricted type are shown to separate partially in Cartesian or cylindrical or spherical coordinates.

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1. Introduction

The aim of this article is to extend results related to separation of variables for flat Lie algebraic Schrödinger operators. Originally, in [11], Alexander Turbiner conjectured the following:

Conjecture 1 (*Turbiner*). *In $\mathbb{R}^2$ there exist no quasi-exactly solvable or exactly solvable problems containing the Laplace–Beltrami operator with flat-space metric tensor, which are characterized by non-separable variables.*

This conjecture was reformulated in more geometrical terms by Rob Milson. The conjecture, which the work in this paper is based on, now reads as follows.

Conjecture 2 (*Turbiner, Second Version*). *Let H be a Lie algebraic Schrödinger operator defined on a two-dimensional manifold. If the symbol of H engenders a Euclidean geometry, i.e. if the corresponding Gaussian curvature is zero, then the spectral equation $H\psi = E\psi$ can be solved by a separation of variables.*

This conjecture is false in general. A counter-example is given by Rob Milson in [6] and [7], together with a proof of a modified version of the conjecture. By adding two extra assumptions, namely an imprimitive action and a compactness requirement, one can prove that the spectral equation can be solved by separation of variables. Furthermore, the imprimitivity hypothesis implies even more than expected: separation will occur in either a Cartesian or a polar coordinate system.

In this paper, it is shown that, in three dimensions, the original conjecture is also false and a proof of a modified version of the 3D-Turbiner's conjecture is given. Again the compactness requirement is indispensable, a condition

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related to the imprimitivity of the action is necessary and a third condition, related to the contravariant metric, will be imposed. In three dimensions, the invariant foliation of an imprimitive action can be a family of curves or a family of surfaces. In the proof of our result, the leaves of the foliation need to be surfaces and such an imprimitive action will be called 2-imprimitive. Like for the two-dimensional case, the 2-imprimitivity of the action will ensure that separation, here only partial, will occur in a Cartesian, cylindrical or spherical coordinate system.

The proofs of both modified versions of the conjecture are based on the following ideas. First, the imprimitive action, which is 2-imprimitive in three dimensions, induces an invariant foliation Λ, for which the leaves are hypersurfaces and will be denoted by $\{\lambda = \text{constant}\}$. The Schrödinger operator $\mathcal{H}$ is Lie algebraic; thus, it is an element of the enveloping algebra of a Lie algebra of first-order differential operators. When applied to λ, the elements of the generating Lie algebra must give back functions of λ. The operator $\mathcal{H}$ will enjoy the same property, that is $\mathcal{H}(\lambda) = f(\lambda)$. Combining the fact that the operator is Lie algebraic with the imprimitivity of the action, one can prove that the leaves of $\Lambda^{\perp}$, the foliation which is perpendicular to Λ, are necessarily geodesics. Then, one of the key tools, the Tiling Theorem, gives a global map from a Euclidean space to our manifold. Thus on pulling back the operator, the leaves of the perpendicular foliation are straight lines. Note that the Tiling Theorem follows from an intermediate one: the Trapping Theorem.

In this setting, one can show that the invariant leaves can only be prescribed curves or surfaces. In two dimensions the curves need to be straight lines or concentric circles, while in three dimensions the surfaces have to be planes, cylinders or spheres. In this context, λ will be either a Cartesian or a radial coordinate. Finally, using the appropriate coordinate system, one checks that the equation $\mathcal{H}\psi = E\psi$ separates with respect to the coordinate λ.

Despite the fact that the path followed to prove the modified 3D-Turbiner's conjecture is similar to the one given in [7], there are several important issues, which were absent in the two-dimensional case, and which appear in our study. We had first to select the appropriate flatness criteria, while in two dimensions there is no such choice. In order to prove the 3D-Trapping Theorem, one has to impose that the diagonal terms of the Ricci curvature tensor be zero. For the 3D-Tiling Theorem, it is the Riemannian curvature tensor that needs to vanish. For this proof of the 3D-Trapping and Tiling Theorems, we have to assume that either the metric can be diagonalized, or that **M** is a transverse, type changing manifold. For the first case, to conclude both theorems, an extra requirement of genericity of the contravariant metric needs to be added. This requirement is related to the non-invertible factors of its components and will be defined later. For the transverse, type changing manifold, we will see that such metric can always be diagonalized and is necessarily generic.

The determination of the possible foliations requires a different approach. While in two dimensions one could only consider the possible foliations by straight lines to conclude, in three dimensions one has to keep in mind the entire picture of the two perpendicular foliations in order to determine the three types of leaves. The arguments are not sophisticated but the proof is long enough for us to devote an entire section to it. Another distinction from the two-dimensional case is the fact that separation of variables is only partial. Indeed, as in the work of Rob Milson, one can isolate one variable, λ, but we are left, in three dimensions, with two variables for which nothing can be said.

In Section 2, we briefly describe the context of Turbiner's conjecture. We define all the notions employed in this paper and we give an example that illustrates how the imprimitivity of the action induces separation of variables. Section 3 fills in the gaps needed to generalize the proof of both 3D-Tiling and 3D-Trapping Theorems. The proofs of these two theorems are omitted since, once this work done, both generalizations are straightforward. In Section 4, we show that, after pulling back the metric to $\mathbb{R}^3$, the only possible leaves of the foliation are planes, cylinders and spheres. This fourth section, involving a succession of simple ad hoc arguments, is crucial for its consequences although the proof itself may be skipped at first reading. Using the results exhibited in the two preceding sections, Section 5 is devoted to the proof of the 3D modified conjecture. Finally a counter-example to the three-dimensional general form of Turbiner's conjecture is exhibited in Section 6.

We conclude by noting that there are deep connections between separation of variables, exact solvability and superintegrability; read for instance [9,3]. However, these lie outside the scope of our paper.

2. General setting

In this section, we introduce the framework and the notions necessary to prove the three-dimensional version of the modified Turbiner's conjecture. The two-dimensional version, see [7,6] for a complete proof, is also discussed.