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Non-uniqueness of the natural and projectively equivariant quantization

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Abstract

In [C. Duval, V. Ovsienko, Projectively equivariant quantization and symbol calculus: Noncommutative hypergeometric functions, Lett. Math. Phys. 57 (1) (2001) 61–67], the authors showed the existence and the uniqueness of a $sl(m+1,\mathbb{R})$-equivariant quantization in non-critical situations. The curved generalization of the $sl(m+1,\mathbb{R})$-equivariant quantization is the natural and projectively equivariant quantization. In [M. Bordemann, Sur l'existence d'une prescription d'ordre naturelle projectivement invariante (submitted for publication). math.DG/0208171] and [Pierre Mathonet, Fabian Radoux, Natural and projectively equivariant quantizations by means of Cartan connections, Lett. Math. Phys. 72 (3) (2005) 183–196], the existence of such a quantization was proved in two different ways. In this paper, we show that this quantization is not unique.

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1. Introduction

A quantization can be defined as a linear bijection from the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on a manifold M (also called the space of *Symbols*) to the space $\mathcal{D}_{\frac{1}{2}}(M)$ of differential operators acting between half-densities.

It is known that there is no natural quantization procedure. In other words, the spaces of symbols and of differential operators are not isomorphic as representations of $\mathrm{Diff}(M)$.

The idea of equivariant quantization, introduced by Lecomte and Ovsienko in [5], is to reduce the group of local diffeomorphisms in the following way.

They considered the case of the projective group $PGL(m+1,\mathbb{R})$ acting locally on the manifold $M=\mathbb{R}^m$ by linear fractional transformations. They showed that the spaces of symbols and of differential operators are canonically

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isomorphic as representations of $PGL(m+1,\mathbb{R})$ (or its Lie algebra $sl(m+1,\mathbb{R})$). In other words, they showed that there exists a unique *projectively equivariant quantization*. In [3], the authors generalized this result to the spaces $\mathcal{D}_{\lambda\mu}(\mathbb{R}^m)$ of differential operators acting between λ- and μ-densities and to their associated graded spaces $\mathcal{S}_\delta$. They showed the existence and uniqueness of a projectively equivariant quantization, provided the shift value $\delta = \mu - \lambda$ does not belong to a set of critical values.

The problem of the $sl(m+1,\mathbb{R})$-equivariant quantization on $\mathbb{R}^m$ has a counterpart on an arbitrary manifold M. In [6], Lecomte conjectured the existence of a quantization procedure depending on a torsion-free connection, that would be natural (in all arguments) and that would remain invariant by a projective change of connection.

After the proof of the existence of such a *Natural and equivariant quantization* given by Bordemann in [1], we analysed in [7] the problem of this existence using Cartan connections. After these works, the question of the uniqueness of this quantization was not yet approached. The uniqueness of the $sl(m+1,\mathbb{R})$-equivariant quantization in the non-critical situations did not imply the uniqueness of the solution in the curved case. The aim of this paper is to show that this solution is not unique, even in non-critical situations, using the theory of Cartan connections.

2. Fundamental tools

For the sake of completeness, we briefly recall in this section the main notions and results of [7]. We refer the reader to this reference or to [4] for additional information. Throughout this note, we denote by M a smooth, Hausdorff and second countable manifold of dimension m.

2.1. Natural and projectively equivariant quantization

Denote by $\mathcal{F}_\lambda(M)$ the space of smooth sections of the vector bundle of λ-densities.

We denote by $\mathcal{D}_{\lambda,\mu}(M)$ the space of differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$ and by $\mathcal{D}^k_{\lambda,\mu}$ the space of differential operators of order at most k. If $\delta = \mu - \lambda$, the associated space of *symbols* will be called $\mathcal{S}^k_\delta(M)$ and σ will represent the *principal symbol operator* from $\mathcal{D}^k_{\lambda,\mu}(M)$ to $\mathcal{S}^k_\delta(M)$.

In these conditions, a *quantization* on M is a linear bijection Q_M from the space of symbols $\mathcal{S}_\delta(M)$ to the space of differential operators $\mathcal{D}_{\lambda,\mu}(M)$ such that

$$\sigma(Q_M(S)) = S, \quad \forall S \in \mathcal{S}^k_\delta(M), \forall k \in \mathbb{N}.$$

A *natural quantization* is a quantization which depends on a torsion-free connection and commutes with the action of diffeomorphisms.

More explicitly, if ϕ is a local diffeomorphism from M to N, then one has

$$Q_M(\phi^*\nabla)(\phi^* S) = \phi^*(Q_N(\nabla)(S)), \quad \forall \nabla \in \mathcal{C}_N, \forall S \in \mathcal{S}_\delta(N).$$

A quantization Q_M is *projectively equivariant* if one has $Q_M(\nabla) = Q_M(\nabla')$ whenever ∇ and ∇' are projectively equivalent torsion-free linear connections on M.

2.2. Projective structures and Cartan projective connections

We consider the group $G = PGL(m+1,\mathbb{R})$ acting on the projective space. We denote by H its isotropy subgroup at the origin. The group H is the semi-direct product $G_0 \rtimes G_1$, where G_0 is isomorphic to $GL(m,\mathbb{R})$ and G_1 is isomorphic to $\mathbb{R}^{m*}$. The Lie algebra associated with H is $\mathfrak{g}_0 \oplus \mathfrak{g}_1$. The Lie algebra associated with G is then equal to $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ is an abelian Lie subalgebra of $\mathfrak{g}$.

We recall that H can be seen as a subgroup of the group of 2-jets G^2_m.

A *projective structure on* M is then a reduction of the second-order frame bundle P^2M to the group H.

The following result [4, p. 147] is the starting point of our method:

Proposition 1 (*Kobayashi–Nagano*). *There is a natural one to one correspondence between the projective equivalence classes of torsion-free linear connections on M and the projective structures on M.*