Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 58 (2008) 122–132
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Equilibrium theory in 2D Riemann manifold for heterogeneous biomembranes with arbitrary variational modes

Yajun Yin*, Jie Yin, Cunjing Lv

Department of Engineering Mechanics, School of Aerospace, FML, Tsinghua University, 100084, Beijing, China

Received 2 February 2007; received in revised form 16 September 2007; accepted 8 October 2007
Available online 12 October 2007

Abstract

Based on the first and second gradient operators and their integral theorems in 2D Riemann manifold, the equilibrium differential equations and geometrically constraint equations for heterogeneous biomembranes with arbitrary variation modes are developed. Through the combination of these equations, the equilibrium theory for heterogeneous biomembranes is established in 2D Riemann manifold. From the equilibrium theory, various interesting information is revealed: Different from homogeneous biomembranes, heterogeneous one posses new equations within the membrane's tangential planes, i.e. the in-plane equilibrium differential equations, the in-plane boundary conditions and the in-plane geometrically constraint equations. Different from the equilibrium theory in Euclidean space, the one in 2D Riemann manifold displays strict constraints between the physical coefficients and characteristic geometric parameters of biomembranes.

MSC: 20F65; 62P35; 70G45; 74L15

Keywords: Heterogeneous biomembranes; Differential operators; Equilibrium differential equations; Geometrically constraint equation; Arbitrary variational mode

1. Introduction

This paper deals with the equilibrium theory for heterogeneous biomembranes with arbitrary variational mode. As far as the equilibrium of biomembranes is concerned, variational principles [1–4] are used to derive the equilibrium differential equations and boundary conditions. During the variational process, how to select the variational modes is a core problem. Theoretically, the most general mode for the variation of the location (i.e. the virtual displacement vector $\boldsymbol{V}$) at a point on a biomembrane should be taken as (see Fig. 1):

$$\boldsymbol{V} = \boldsymbol{v} + \psi \boldsymbol{n}. \tag{1}$$

Here $\boldsymbol{v}$ is the tangential displacement vector and $\psi \boldsymbol{n}$ is the normal one with $\boldsymbol{n}$ the unit outward normal vector of the biomembrane. In the past, most researches mainly deal with the normal deformation mode. Recently, both normal and tangential deformation modes (i.e. arbitrary variational modes) have drawn the attentions of researchers [5,6].

* Corresponding author. Tel.: +86 10 62795536; fax: +86 10 62781824.
E-mail address: yinyj@mail.tsinghua.edu.cn (Y. Yin).

0393-0440/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2007.10.001

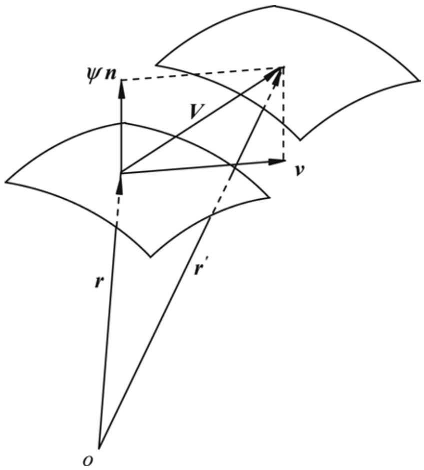

Fig. 1. A schematic diagram for the arbitrary deformation mode in a biomembrane.

This paper will concentrate on heterogeneous biomembranes with arbitrary variational modes. This topic is worth to be explored systematically because of the following reasons: First, practical biomembranes are usually heterogeneous. For example, cell membranes consist of various constituents such as lipid molecules, proteins and enzymes, etc. Interactions among them may cause the aggregations of constituents and lead to domains with certain biological functions. Second, the roles played by tangential deformation mode may be more important in heterogeneous biomembranes than in homogeneous one. In fact, the theory for heterogeneous biomembranes with tangential deformation mode included may bring about new information. In short, the theory for heterogeneous biomembranes may have its own characteristics and should be investigated independently.

The equilibrium theory in this paper will be specially expressed through differential operators. The reason is as follows. Once the tangential deformation mode is introduced, the theoretical system may become much more complicated. To simplify the theory and make the theory more understandable to researchers, a simple and convenient mathematical frame is necessary. This may be realized through differential operators. Recently, new gradient operator is defined from the studies on biomembranes and a series of integral theorems in 2D Riemann manifold are proved [7–9]. These mathematical results in turn may provide powerful tools to the explorations of heterogeneous biomembranes either with normal deformation mode [10] or with arbitrary one.

This paper includes four parts. First, a few differential operators are introduced. Second, these operators and their integral theorems are applied to heterogeneous biomembranes to derive the equilibrium equations and geometrically constraint equations. To one's surprise, new equations, including the equilibrium differential equations, boundary conditions and geometrically constraint equations within the tangential plane of a biomembrane are revealed. Third, the potential importance and possible applications of the in-plane equations are predicted. Fourth, two appendixes in which all the variational quantities are expressed through differential operators are presented.

2. Brief summary of a few differential operators

In physics and mechanics, gradient is an important concept. A gradient is physically a "force" that drives various dynamics in macro or micro scales. Without pressure gradient, deformation gradient, temperature gradient and electromagnetic gradient, there would be no fluid dynamics, solid mechanics, thermal dynamics and electromagnetism. In differential geometry, there is the conventional 2D gradient operator ∇ defined on a curved surface:

$$\nabla = g^{ij}\boldsymbol{g}_i \frac{\partial}{\partial u^j} \quad (i, j = 1, 2). \tag{2}$$

Recently, another 2D gradient operator $\overline{\nabla}$ is derived during the study on biomembranes [7–9]:

$$\overline{\nabla} = \hat{L}^{ij}\boldsymbol{g}_i \frac{\partial}{\partial u^j} \quad (i, j = 1, 2). \tag{3}$$