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# A counterexample to a conjecture due to Douglas, Reinbacher and Yau

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**Abstract**

In "Branes, Bundles and Attractors: Bogomolov and Beyond", by Douglas, Reinbacher and Yau, the authors state the following conjecture: Consider a simply connected surface $X$ with ample or trivial canonical line bundle. Then, the Chern classes of any stable vector bundle with rank $r \geq 2$ satisfy $2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) \geq 0$. The goal of this short note is to provide two sources of counterexamples to this strong version of the Bogomolov inequality.


*MSC:* 14F05

*Keywords:* Chern classes; Stable; Vector bundles

## 1. Introduction

In [2] Douglas, Reinbacher and Yau state several conjectures arising from the attractor mechanism in type II string theory concerning possible Chern classes of stable vector bundles on algebraic varieties. In particular, the paper contains the following conjecture which is a slight strengthening of the Bogomolov inequality.

**Conjecture 1.1.** *Consider a simply connected surface $X$ with ample or trivial canonical bundle and let $H$ be an ample line bundle on $X$. Then, the Chern classes of any $\mu_H$-stable vector bundle of rank $r \geq 2$ satisfy*

$$2rc_2 - (r-1)c_1^2 - \frac{r^2}{12}c_2(X) \geq 0.$$

On the basis of physical evidence, this conjecture was first stated for Kähler manifolds of dimension $n$ in a preliminary version of Douglas, Reinbacher and Yau's paper. In [5] Jardim provides examples that show that the conjecture does not hold for stable vector bundles on Calabi–Yau threefolds. In a revised version of the paper of Douglas, Reinbacher and Yau, the original conjecture was replaced by the above statement concerning the Chern classes of stable vector bundles on simply connected surfaces with ample or trivial canonical bundle. The goal of this

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paper is to prove that the reformulated version is also false. We will provide two kinds of examples. The first one (see Proposition 2.1) concerns rank $r \geq 2$ vector bundles on a generic $K3$ surface $X$ (i.e. on a generic algebraic surface $X$ with $q(X) = 0$ and trivial canonical line bundle). The second one (see Proposition 3.2) is devoted to rank $r \geq 3$ vector bundles on a surface $X$ in $\mathbb{P}^3$ of degree $d \geq 7$ (and hence its canonical line bundle is ample).

*Terminology:* Let $H$ be an ample line bundle on a smooth projective algebraic surface $X$. For a torsion free sheaf $F$ on $X$ we set

$$\mu(F) = \mu_H(F) := \frac{c_1(F)H}{rk(F)}.$$

The sheaf $F$ is said to be $\mu_H$*-semistable* if

$$\mu_H(E) \leq \mu_H(F)$$

for all non-zero subsheaves $E \subset F$ with $rk(E) < rk(F)$; if strict inequality holds then $F$ is $\mu_H$*-stable*. Notice that for rank $r$ vector bundles $F$ on $X$ with $(c_1(F)H, r) = 1$, the concepts of $\mu_H$-stability and $\mu_H$-semistability coincide.

Recall that for any rank $r$ vector bundle $F$ on a cyclic variety $X$ with $\mathrm{Pic}(X)$ generated by $h$, there is a uniquely determined integer $k_F$ such that if $c_1(F(k_F h)) = c_1 h$, then $-r + 1 \leq c_1 \leq 0$. We set $F_{\mathrm{norm}} = F(k_F h)$.

## 2. First example

The goal of this section is to see that Conjecture 1.1 fails for $\mu_H$-stable rank 2 vector bundles on generic $K3$ surfaces with trivial canonical line bundle. To this end, let $X$ be a complex algebraic $K3$ surface, that is $X$ is a complete regular surface with trivial canonical line bundle and irregularity $q(X) = 0$. According to [7], we will say that a vector bundle $E$ on $X$ is *exceptional* if

$$\dim \mathrm{Hom}(E, E) = 1 \quad \text{and} \quad \mathrm{Ext}^1(E, E) = 0,$$

i.e., $E$ is simple and rigid. Any coherent sheaf $F$ on $X$ has associated a *Mukai vector*

$$v(F) = \left(r, c_1, \frac{c_1^2}{2} + r - c_2\right)$$

where $r = \mathrm{rank}(F)$ and $c_1, c_2$ denote the first and second Chern classes of $F$. A Mukai vector is called exceptional if according to the inner product defined in the Mukai lattice (see [8]) the following equality holds:

$$v(F)^2 = c_1^2 - 2r\left(r - c_2 + \frac{c_1^2}{2}\right) = -2.$$

When $X$ is a $K3$ surface, $c_2(X) = 24$. Indeed we have

$$2 = \chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + c_2(X)) = \frac{c_2(X)}{12}.$$

So, in that case Conjecture 1.1 is equivalent to saying that the Chern classes of any $\mu_H$-stable vector bundle of rank $r \geq 2$ satisfy

$$2rc_2 - (r-1)c_1^2 - 2r^2 \geq 0.$$

Let us see that there exist infinitely many $\mu_H$-stable vectors on a generic $K3$ surface whose Chern classes do not satisfy the inequality above. First of all notice that if $X$ is a generic $K3$ surface then $\mathrm{Pic}(X) \cong \mathbb{Z}$.

**Proposition 2.1.** *Let $X$ be a generic $K3$ surface and let $H$ be an arbitrary ample line bundle on $X$. For any Mukai vector $v = (r, c_1, \frac{c_1^2}{2} + r - c_2)$ such that $(r, c_1 H) = 1$ and*

$$2rc_2 - (r-1)c_1^2 = 2r^2 - 2$$

*there exists a $\mu_H$-stable rank $r$ vector bundle $E$ on $X$ with Mukai vector $v(E) = v$.*