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Energy–momentum tensor on foliations

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Abstract

In this paper, we give a new lower bound for the eigenvalues of the Dirac operator on a compact spin manifold. This estimate is motivated by the fact that in its limiting case a skew-symmetric tensor (see Eq. (1.6)) appears that can be identified geometrically with the O'Neill tensor of a Riemannian flow, carrying a transversal parallel spinor. The Heisenberg group which is a fibration over the torus is an example of this case. Sasakian manifolds are also considered to be particular examples of Riemannian flows. Finally, we characterize the 3-dimensional case by a solution of the Dirac equation.

MSC: 53C27; 53C12; 53C25; 57C30

Keywords: Dirac operator; Energy–momentum tensor; Hypersurfaces; Second fundamental form; Riemannian flows; O'Neill tensor

1. Introduction

The study of the spectrum of the Dirac operator defined on a spin manifold M, has been intensively investigated since it contains subtle information on the geometry of the manifold. In [10], Friedrich proved that on a compact spin manifold M of dimension n, the first eigenvalue λ of D_M satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M \mathrm{Scal}_M, \tag{1.1}$$

where Scal_M is the scalar curvature of M, supposed to be positive. The proof is based on the modification of the Levi-Civita connection of the spinor bundle in the direction of the identity and the use of the Schrödinger–Lichnerowicz formula [21]. The limiting case of (1.1) is characterized by the existence of a special section of the spinor bundle called *Killing spinor* satisfying an overdetermined differential equation. The manifold is in that case Einstein.

Observe that Friedrich's estimate contains no information for manifolds with negative or vanishing scalar curvature. Hence the estimate is established in [15] for all manifolds (the scalar curvature could be negative) where the author modified the Levi-Civita connection in the direction of a symmetric tensor leading to a lower bound of the spinorial Laplacian by the norm squared of this tensor.

In fact, we suppose that on a spin manifold M, there exists a spinor field Ψ such that it satisfies for all $X \in \Gamma(TM)$,

$$\nabla^M_X \Psi = -E(X) \cdot \Psi, \tag{1.2}$$

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where E is a symmetric 2-tensor defined on TM. Then with the properties of Clifford multiplication, we see that E is equal to the tensor T^{Ψ}, called the *energy–momentum tensor*, defined on the complement set of zeroes of Ψ for all $X, Y \in \Gamma(TM)$ by

$$T^{\Psi}(X,Y) = \frac{1}{2}\Re\left(X \cdot \nabla^M_Y \Psi + Y \cdot \nabla^M_X \Psi, \frac{\Psi}{|\Psi|^2}\right). \tag{1.3}$$

Hence he proved that for any eigenspinor Ψ of D_M associated with the first eigenvalue λ, we have

$$\lambda^2 \geq \inf_M \left(\frac{\mathrm{Scal}_M}{4} + |T^{\Psi}|^2\right). \tag{1.4}$$

The important point is that the set of zeroes of Ψ has a Hausdorff dimension equal to $n-2$ (see [1]) and hence its measure is zero. The estimate (1.4) improves Friedrich's inequality since by the Cauchy–Schwarz inequality, $|T^{\Psi}|^2 \geqslant \frac{(\mathrm{tr}(T^{\Psi}))^2}{n}$ (here tr denotes the trace). The existence of a spinor field satisfying, for all $X \in \Gamma(TM)$ the equation $\nabla^M_X \Psi = -T^{\Psi}(X) \cdot \Psi$, characterizes its limiting case. In this case, it is not easy to describe geometrically such manifolds since the lower bound of (1.4) depends on the eigenspinor in question.

The study of Eq. (1.2) in extrinsic spin geometry is the key point for a natural interpretation of this tensor. If the dimension of M is equal to 2, Friedrich [11] proved that the existence of a spinor field Ψ, with constant norm satisfying $D_M \Psi = f\Psi$, where f is a real function on M, is equivalent to the existence of a pair (Ψ, E) satisfying (1.2), where E is a symmetric tensor of trace f. This also implies that E is a Gauss–Codazzi tensor and the manifold M is locally immersed into the Euclidean space $\mathbb{R}^3$ with a mean curvature equal to f. Here we have the following fact [22]: If M^n is a hypersurface of a manifold N, carrying a parallel spinor, then the energy–momentum tensor appears naturally as the second fundamental form h of the hypersurface. Moreover, if the mean curvature H is a positive constant, then we are in the limiting case of the extrinsic estimate established in [16] and we have

$$\frac{n^2H^2}{4} = \frac{\mathrm{Scal}_M}{4} + |T^{\Psi}|^2 = \frac{\mathrm{Scal}_M}{4} + \frac{|h|^2}{4}.$$

In this paper, we study Eq. (1.2) in a general case. We assume that on a Riemannian spin manifold (M, g_M), there exists a spinor field Ψ satisfying, for all $X \in \Gamma(TM)$, the equation

$$\nabla^M_X \Psi = -E(X) \cdot \Psi, \tag{1.5}$$

where E is *any* endomorphism of TM. By using the properties of Clifford multiplication, we find that the symmetric part of E is T^{Ψ} and the skew-symmetric part of E is the tensor defined, on the complement set of zeroes of Ψ, by

$$Q^{\Psi}(X,Y) = \frac{1}{2}\Re\left(Y \cdot \nabla^M_X \Psi - X \cdot \nabla^M_Y \Psi, \frac{\Psi}{|\Psi|^2}\right), \tag{1.6}$$

for all $X, Y \in \Gamma(TM)$ (see Section 2). The problem here is to relate these two tensors to the spectrum of the Dirac operator. We prove that if we modify the Levi-Civita connection in the direction of these two tensors, the spinorial Laplacian is bounded from below by the sum of the norm squared of these two tensors. Thus we have:

Theorem 1.1. *Let (M, g_M) be a compact spin manifold, then the first eigenvalue of the Dirac operator satisfies*

$$\lambda^2 \geq \inf_M \left(\frac{\mathrm{Scal}_M}{4} + |T^{\Psi}|^2 + |Q^{\Psi}|^2\right), \tag{1.7}$$

where Ψ is an eigenspinor of D_M^2 associated with λ^2. The equality case of (1.7) is characterized by a solution of (1.5).

The Heisenberg group Nil_3 and the solvable group Sol_3 are examples of limiting manifolds with negative scalar curvature (the term Q^{Ψ} is equal to zero, see Examples 1 and 2), so also is the Riemannian product $\mathbb{S}^1 \times \mathbb{S}^2$ with positive scalar curvature (the term T^{Ψ} is equal to zero, see Example 3).

The study of foliations and in particular the transverse geometry of *Riemannian flows* [8], which are locally given by Riemannian submersions with 1-dimensional fibres, will allow for a better understanding of the tensor Q^{Ψ}. In fact,