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Fedosov's formal symplectic groupoids and contravariant connections

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Abstract

Using Fedosov's approach we give a geometric construction of a formal symplectic groupoid over any Poisson manifold endowed with a torsion-free Poisson contravariant connection. In the case of Kähler–Poisson manifolds this construction provides, in particular, the formal symplectic groupoids with separation of variables. We show that the dual of a semisimple Lie algebra does not admit torsion-free Poisson contravariant connections.

MSC: primary 22A22; secondary 53D05

Keywords: Symplectic groupoids; Poisson manifolds; Contravariant connections; Deformation quantization

1. Introduction

A symplectic groupoid over a Poisson manifold M is a symplectic manifold Σ endowed with a partially defined multiplication and the source, target, inverse, and unit mappings satisfying several axioms. In particular, the source and the target mappings are a Poisson and an anti-Poisson mapping from Σ to M, respectively. Symplectic groupoids play the rôle of semiclassical counterparts of associative algebras treated as quantum objects. Symplectic groupoids were introduced independently by Karasëv [11], Weinstein [14,3], and Zakrzewski [16]. There is a corresponding notion of a formal symplectic groupoid on the formal neighborhood (Σ, Λ) of a

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Lagrangian submanifold Λ of a symplectic manifold Σ whose principal example is the formal neighborhood (Σ, Λ) of the Lagrangian unit space Λ of a symplectic groupoid on a symplectic manifold Σ (see [10]). Formal symplectic groupoids were first introduced in [2] in terms of formal generating functions of their (formal) Lagrangian product spaces. It was shown in [10] that to each natural deformation quantization on a Poisson manifold M there corresponds a canonical formal symplectic groupoid on (T^*M, Z), where Z is the zero section of T^*M. The main result of [2] is the description of the formal symplectic groupoid of Kontsevich deformation quantization. The formal symplectic groupoid of Fedosov's star-product was described in [9]. This paper is motivated by the following observation. On the one hand, it is known that deformation quantizations with separation of variables (also known as deformation quantizations of the Wick type, see [7] and [1]) are a particular case of Fedosov's deformation quantizations (see [12]). On the other hand, it was shown in [10] that the corresponding formal symplectic groupoids "with separation of variables" can be naturally extended from Kähler manifolds to Kähler–Poisson manifolds, while it is impossible to extend the star-products with separation of variables to the Kähler–Poisson manifolds in a naive direct way (see [8]). In this paper we show that the construction of the formal symplectic groupoids of Fedosov's deformation quantizations from [9] can be naturally extended to the Poisson manifolds endowed with a torsion-free Poisson contravariant connection. We call the formal symplectic groupoids obtained via this construction Fedosov's formal symplectic groupoids.

On a Kähler–Poisson manifold, there is a natural torsion-free Poisson contravariant connection which we call the Kähler–Poisson contravariant connection. We show that Fedosov's formal symplectic groupoid constructed with the use of the Kähler–Poisson contravariant connection is a formal symplectic groupoid with separation of variables.

Any symplectic manifold admits symplectic (torsion-free) connections and therefore Poisson torsion-free contravariant connections. However, this is not the case for general Poisson manifolds. We prove that the dual space of a semisimple Lie algebra does not admit a torsion-free Poisson contravariant connection.

2. Linear contravariant connections

Contravariant derivatives were introduced by Vaisman in [13]. The corresponding notion of a contravariant connection was extensively studied by Fernandes in [5].

Let M be a Poisson manifold endowed with the Poisson bivector field Π. Then the Poisson bracket of functions $f, g \in C^\infty(M)$ is given by

$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g).$$

Define a bundle map $\# : T^*M \to TM$ by the formula

$$\langle \beta, \#\alpha \rangle = \Pi(\alpha, \beta),$$

where $\alpha, \beta \in \Omega^1(M)$ are 1-forms on M and $\langle \cdot, \cdot \rangle$ is the natural pairing of T^*M and TM. It is known that on the space $\Omega^1(M)$ of 1-forms on M there is a Lie bracket

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - \mathrm{d}\Pi(\alpha, \beta),$$

where $\alpha, \beta \in \Omega^1(M)$ and $\mathcal{L}$ denotes the Lie derivative. If $\alpha = \mathrm{d}f$ and $\beta = \mathrm{d}g$ for $f, g \in C^\infty(M)$, then

$$[\mathrm{d}f, \mathrm{d}g] = \mathrm{d}\{f, g\}. \tag{1}$$