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Holomorphic spectrum of twisted Dirac operators on compact Riemann surfaces

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Abstract

Given a Hermitian line bundle L with a harmonic connection over a compact Riemann surface (S, g) of constant curvature, we study the spectral geometry of the corresponding twisted Dirac operator $\mathcal{D}$. This problem is analyzed in terms of the natural holomorphic structures of the spinor bundles $\mathbb{E}^{\pm}$ defined by the Cauchy–Riemann operators associated with the spinorial connection. By means of two elliptic chains of line bundles obtained by twisting $\mathbb{E}^{\pm}$ with the powers of the canonical bundle K_S, we prove that there exists a certain subset $\mathrm{Spec}_{hol}(\mathcal{D})$ of the spectrum such that the eigensections associated with $\lambda \in \mathrm{Spec}_{hol}(\mathcal{D})$ are determined by the holomorphic sections of a certain line bundle of the elliptic chain. We give explicit expressions for the holomorphic spectrum and the multiplicities of the corresponding eigenvalues according to the genus p of S, showing that $\mathrm{Spec}_{hol}(\mathcal{D})$ does not depend on the spin structure and depends on the line bundle L only through its degree. This technique provides the whole spectrum of $\mathcal{D}$ for genus $p = 0$ and 1, whereas for genus $p > 1$ we obtain a finite number of eigenvalues.

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1. Introduction

Hitchin proved in [16] that there exists a one-to-one correspondence between the spin structures of a complex Hermitian manifold X and the holomorphic square roots $K_X^{\frac{1}{2}}$ of

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its canonical line bundle. If in addition X is Kähler, then the twisted Dirac operator $\mathcal{D}$ associated with a Hermitian holomorphic vector bundle E is completely determined by the Cauchy–Riemann operator $\bar{\partial}$ of $E \otimes K_X^{\frac{1}{2}}$ and we get explicitly $\mathcal{D} = \sqrt{2}(\bar{\partial}^* + \bar{\partial})$ (see for instance [4]). Therefore one may expect a close relationship between the holomorphic geometry of $E \otimes K_X^{\frac{1}{2}}$ and the spectral geometry of $\mathcal{D}$. In fact, by means of Hodge theory, one proves that the kernel of $\mathcal{D}$ on a compact Kähler spin manifold can be identified with the cohomology groups $H^\bullet\left(X, \mathcal{O}_X\left(E \otimes K_X^{\frac{1}{2}}\right)\right)$. This identification has been used by several authors to compute the harmonic spinors of certain Kähler manifolds, see for instance [1,5,8,16,19]. However, apart from the study of the kernel of the Dirac operator and some geometric estimates for the first eigenvalue [7,18,20], the holomorphic techniques have been used very little in the explicit computation of the spectrum of this operator.

One of the aims of this paper is to show that, under suitable conditions, it is possible to extend the above mentioned relationship to eigenvalues different from zero. In the present paper we restrict our attention to the particular case of a compact Riemann surface S endowed with a Riemannian metric of constant curvature and the twisting bundle is assumed to be a line bundle endowed with a harmonic connection. With these assumptions, we are able to prove that there exists a subset $\mathrm{Spec}_{hol}(\mathcal{D})$ of the spectrum of the twisted Dirac operator $\mathcal{D}$ which can be explicitly computed by means of holomorphic techniques. Moreover, for every $\lambda \in \mathrm{Spec}_{hol}(\mathcal{D})$, the corresponding eigensections can be identified with the holomorphic sections of a line bundle belonging to an elliptic chain defined over S. Due to these facts we call $\mathrm{Spec}_{hol}(\mathcal{D})$ the holomorphic spectrum of $\mathcal{D}$.

Let us explain now our method for analyzing $\mathrm{Spec}_{hol}(\mathcal{D})$ for a compact Riemann surface.

Any compact Riemannian surface (S, g) of genus p has 2^{2p} non-equivalent spin structures. If we fix one of them, then the spinor bundle is the $\mathbb{Z}_2$-graded vector bundle $\mathbb{S} = \Lambda^{0,\bullet}T^*(S) \otimes K_S^{\frac{1}{2}} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Given a Hermitian line bundle L endowed with a harmonic connection ∇, the associated twisted Clifford module is $\mathbb{E} = \mathbb{S} \otimes L = \mathbb{E}^+ \oplus \mathbb{E}^-$. The twisted Dirac operator is odd with respect to this $\mathbb{Z}_2$-gradation, therefore we write $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$. It is well known that the spectral resolution of $\mathcal{D}$ is completely determined by its kernel and the spectral resolution of $\mathcal{D}^-\mathcal{D}^+$ or $\mathcal{D}^+\mathcal{D}^-$.

We consider two chains of line bundles $\mathcal{C}^\bullet(\mathbb{E}^\pm) = \{K_S^q \otimes \mathbb{E}^\pm\}_{q\in\mathbb{Z}}$. The twisted spinorial connection on $\mathbb{E}^\pm$ and the Levi-Civita connection on K_S^q induce an integrable unitary connection $\nabla_\pm^q$ on each bundle $K_S^q \otimes \mathbb{E}^\pm$. The associated Cauchy–Riemann operators $\partial^{\nabla_\pm^q}$ and $\bar{\partial}^{\nabla_\pm^q}$ are elliptic and define morphisms between the differentiable sections of the bundles of each elliptic chain. For $q = 0$, one has $\mathcal{D}^+ = \sqrt{2}\,\bar{\partial}^{\nabla_+^0}$ and $\mathcal{D}^- = -\sqrt{2}\,\partial^{\nabla_-^0}$. On the elliptic chains we define the Laplacians $\Delta_\pm^q = (\partial^{\nabla_\pm^q})^*\partial^{\nabla_\pm^q}$ and $\Delta_q^\pm = (\bar{\partial}^{\nabla_\pm^q})^*\bar{\partial}^{\nabla_\pm^q}$. Therefore, $\mathcal{D}^-\mathcal{D}^+ = 2\,\Delta_0^+$, $\mathcal{D}^+\mathcal{D}^- = 2\,\Delta_-^0$ and the spectral resolution of $\mathcal{D}^2$ can be expressed in terms of the Laplacians of the elliptic chains.

If (S, g) is a Riemann surface of constant scalar curvature κ then the Kähler identities for $\nabla_\pm^q$ give the following fundamental commutation relations

$$\Delta_{-q}^{\pm}\partial^{\nabla_\pm^{-(q+1)}} - \partial^{\nabla_\pm^{-(q+1)}}\Delta_{-(q+1)}^{\pm} = \left[(q+1)\frac{\kappa}{2} \mp \frac{\kappa}{4} + B\right]\partial^{\nabla_\pm^{-(q+1)}}$$

$$\Delta_\pm^q\bar{\partial}^{\nabla_\pm^{(q+1)}} - \bar{\partial}^{\nabla_\pm^{(q+1)}}\Delta_\pm^{(q+1)} = \left[(q+1)\frac{\kappa}{2} \pm \frac{\kappa}{4} - B\right]\bar{\partial}^{\nabla_\pm^{(q+1)}}$$