Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Geometry and Physics 56 (2006) 754–761
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Homogeneous Lorentzian spaces admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$

P. Meessen*

Physics Department, Theory, CERN, 1211 Geneva 23, Switzerland

Received 13 January 2005; received in revised form 14 April 2005; accepted 26 April 2005
Available online 8 June 2005

Abstract

We show that a homogeneous Lorentzian space admitting a homogeneous structure of type $\mathcal{T}_1 \oplus \mathcal{T}_3$ is either a locally symmetric space or a singular homogeneous plane wave.

Keywords: Homogeneous spaces; Plane waves; 22F30; 83C20

A theorem by Ambrose and Singer [1], generalized to arbitrary signature in [2], states that on a reductive homogeneous space, there exists a metric connection $\bar{\nabla} = \nabla - S$, with ∇ the Levi-Cività connection, that parallelizes the Riemann tensor R, and the $(1, 2)$-tensor S, i.e. $\bar{\nabla} g = \bar{\nabla} R = \bar{\nabla} S = 0$. Since a $(1, 2)$-tensor in $D \geq 3$ decomposes into three irreps of $\mathfrak{so}(D)$, one can classify the reductive homogeneous spaces by the occurrence of one of these irreps in the tensor S [3,4]. This leads to eight different classes, which range from the maximal, denoted by $\mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3$, to the minimal $\{0\}$. Clearly, homogeneous spaces of type $\{0\}$ are just symmetric spaces. Moreover, also the homogeneous spaces admitting a homogeneous structure of type $\mathcal{T}_i$ ($i = 1, 2$ or 3) have been characterized. For the case at hand it is worth knowing that the homogeneous spaces with a $\mathcal{T}_3$ structure, for which S corresponds to a three-form, are naturally reductive spaces [3,4] and that strictly Riemannian

* Tel.: +41 2276 72447.
E-mail address: patrick.meessen@cern.ch.

0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2005.04.016

homogeneous $\mathcal{T}_1$ spaces are locally symmetric spaces [3]. Since a homogeneous structure of type $\mathcal{T}_1$ is defined by an invariant vector field ξ, one must distinguish between two cases in the Lorentzian setting: the non-degenerate case, for which ξ is a space- or time-like vector, and the degenerate case, when ξ is a null vector. In the former case, Gadea and Oubiña [4] proved that, analogously to the strictly Riemannian case, the space is locally symmetric. In the degenerate case, Montesinos Amilibia [5] showed that a homogeneous Lorentzian space admitting a degenerate $\mathcal{T}_1$ structure is a time-independent singular homogeneous plane wave [6]. A small calculation shows that a generic, i.e. time-dependent, singular homogeneous plane wave admits a degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure, see, e.g. Appendix A. (By a (non-)degenerate $\mathcal{T}_1 \oplus \mathcal{T}_3$ structure, we mean that the vector field ξ characterizing the $\mathcal{T}_1$ contribution has (non-)vanishing norm.) This then automatically leads to the question of whether the singular homogeneous plane waves exhaust the degenerate case in the $\mathcal{T}_1 \oplus \mathcal{T}_3$ class. As we will see, the answer is affirmative.

In the $\mathcal{T}_1 \oplus \mathcal{T}_3$ case, the homogeneous structure is given by [3]

$$\bar{\nabla}_X Y - \nabla_X Y = -S_X Y = -T_X Y - g(X, Y)\xi + \alpha(Y)X,$$

where we have defined $\alpha(X) = g(\xi, X)$ and $T_X Y(= -T_Y X)$ is the $\mathcal{T}_3$ contribution. Since the metric and S are parallel under $\bar{\nabla}$ and ξ is the contraction of S, it follows that $\bar{\nabla}\xi = 0$ or, written differently:

$$\nabla_X \xi = T_X \xi + \alpha(X)\xi - \alpha(\xi)X.$$

This equation, together with the fact that T is a three-form, implies that $\nabla_\xi \xi = 0$, i.e. ξ is a geodesic vector.

Given an isometry algebra $\mathfrak{g}$ (i.e. the Lie algebra of a Lie group acting transitively by isometries on a given homogeneous space), with a reductive split $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{h} \subseteq \mathfrak{so}(1, n+1)$ is the isotropy subalgebra, it is possible, and usually done, to identify $\mathfrak{m}$ with $\mathbb{R}^{1,n+1}$; the action of $\mathfrak{h}$ on $\mathfrak{m}$ can then be given by the vector representation of $\mathfrak{so}(1, n+1)$ [7]. This identification enables one to express the algebra in terms of S and the curvature $\bar{R}$ as, limiting ourselves to the $\mathfrak{m} \times \mathfrak{m}$ commutator,

$$[X, Y] = S_X Y - S_Y X + \bar{R}(X, Y), \tag{1}$$

where S and $\bar{R}$ are evaluated at some point p. In the above formula, $\bar{R}$ signals the presence of $\mathfrak{h}$ in $[\mathfrak{m}, \mathfrak{m}]$. From now on, we only consider this Lie algebra and all the relevant tensor fields are evaluated at a specific point, even though this is not stated explicitly.

Up to this point not too much has been said about $\mathfrak{h}$, and in fact not too much can be said. It is known, however [7], that a tensor field parallelized by $\bar{\nabla}$, when evaluated at a point corresponds to an $\mathfrak{h}$-invariant tensor. Since in this article we take ξ (an $\mathfrak{h}$-invariant vector field as $\bar{\nabla}\xi = 0$) to be non-vanishing, this means that $\mathfrak{h} \subseteq \mathfrak{so}(n+1)$ when ξ is light-like, $\mathfrak{h} \subseteq \mathfrak{so}(1, n)$ when ξ is space-like and $\mathfrak{h} \subseteq \mathfrak{iso}(n)$ when ξ is null.

Let us briefly outline the manner in which we arrive at our results: given a reductive homogeneous space with reductive split $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, the subalgebra $\mathfrak{g}' = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}] = \mathfrak{m} + \mathfrak{h}'$ is an ideal of $\mathfrak{g}$. It is this ideal, which is the Lie algebra of a Lie group still acting