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IFFT-equivariant quantizations

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Abstract

The existence and uniqueness of quantizations that are equivariant with respect to conformal and projective Lie algebras of vector fields were recently obtained by Duval, Lecomte and Ovsienko. In order to do so, they computed spectra of some Casimir operators. We give an explicit formula for those spectra in the general framework of *IFFT*-algebras classified by Kobayashi and Nagano. We also define *tree-like* subsets of eigenspaces of those operators in which eigenvalues can be compared to show the existence of IFFT-equivariant quantizations. We apply our results to prove the existence and uniqueness of quantizations that are equivariant with respect to the infinitesimal action of the symplectic (resp. pseudo-orhogonal) group on the symplectic (resp. pseudo-orthogonal) Grassmann manifold.

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1. Introduction

The word “quantization” carries several different meanings, both in physics and mathematics. One approach – see for instance Ref. [12] – is to consider a quantization procedure

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as a linear bijection from the space of *symbols* $\mathrm{Pol}(T^*M)$ – smooth functions on the cotangent bundle of a manifold M that are polynomial along the fibres – to the space $\mathcal{D}_{1/2}(M)$ of linear differential operators acting on half-densities. It is known that these spaces cannot be canonically identified. In other words, there does not exist a preferred quantization procedure.

The concept of *equivariant quantization* was introduced and developed in Refs. [2,10,11]. These recent works take care of the symmetries of the classical situation to quantize.

If G is a group acting on the manifold M, a G-equivariant quantization is an isomorphism of representations of G between the spaces of symbols and of differential operators. Obviously, such an identification does not exist for all groups G acting on M, for instance those spaces are not equivalent as $\mathrm{Diff}(M)$-modules. At the infinitesimal level, if G is a Lie group, its action gives rise to a Lie subalgebra $\mathfrak{g}$ of vector fields over M and one is led to build a $\mathfrak{g}$-equivariant linear bijection. Lecomte and Ovsienko examined the case of a projective structure on a manifold of dimension n, with $G = SL(n+1, R)$ and then, together with Duval, the case of the group $G = SO(p+1, q+1)$ on a manifold of dimension $p+q$. That latter group defines conformal transformations with respect to a pseudo-Riemannian metric.

In these works, the authors consider the more general modules $\mathcal{D}_{\lambda,\mu}$ of differential operators transforming λ-densities into μ-densities. These parameters give rise to the *shift* value $\delta = \mu - \lambda$ and to the special case $\delta = 0$, which can be specialized to the original problem. They obtain existence and uniqueness (up to normalization) results for a quantization procedure in both projective and conformal cases, provided the shift value does not belong to a *critical* set. Furthermore, they show that this set never contains zero.

In suitable charts, the subalgebras mentioned up to now are realized by polynomial vector fields and they share the property of being maximal proper subalgebras of the algebra of polynomial vector fields.

In Ref. [1], we investigated this maximality property and showed that the finite dimensional, graded and maximal proper subalgebras of the Lie algebra of polynomial vector fields over a Euclidean vector space correspond to the list of so called "Irreducible Filtered Lie algebras of Finite Type"(IFFT-algebras), classified by Kobayashi and Nagano in Ref. [7].

Our concern in this paper is to deal with the natural next question : *"Is it possible to build (unique) equivariant quantizations with respect to the IFFT-algebras ?"*

The original construction of the conformally equivariant quantization (see Ref. [2]) involves the computation of the spectrum of the Casimir operator of $so(p+1, q+1)$ acting on the space of symbols. The obstructions to the existence of a quantization show up as equalities among some eigenvalues of that operator. It was also shown in Ref. [2] how the relevant eigenvalues that should be compared are associated to *tree-like subsets* of eigenspaces.

Section 3 of the present article is devoted to this computation. We obtain, for a wide range of IFFT-algebras, a formula where the eigenvalues are expressed in terms of the dimension of the manifold and of the highest weights of some finite dimensional representations of the semisimple part of the linear isotropy algebra of $\mathfrak{g}$ (see Ref. [8]).

In Section 4, we propose a general definition for the above-mentioned tree-like subsets. A few elementary properties of these subspaces allow us to reformulate the existence theorem for equivariant quantizations in the framework of IFFT-algebras.