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A note on symmetric connections

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Abstract

In this paper we analyze a reciprocal of the fundamental theorem of Riemannian geometry. We give a condition for a symmetric connection to be locally the Levi-Civita connection of a metric. We also construct a couple of natural examples of connections on the n-dimensional torus and investigate the global problem.

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1. Introduction

The problem of the determination of a metric compatible with a given symmetric connection ∇ in the tangent bundle TM of a smooth manifold M has some recent history. The problem was investigated mainly by mathematical physicists like Thompson in [5] and [6] and Edgar in [2]) and [3]. In his paper [2] Edgar gives necessary and sufficient conditions for a volume preserving connection (which in our invariant formulation is equivalent with

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the vanishing of the “Chern” form associated to the connection) to be locally the Levi-Civita connection of a metric. In this paper we give an example of a connection on the n-dimensional torus which satisfies the conditions of his theorem, but which is not globally metric. Global natural questions also arise and we address this question by analyzing a natural map. This map can be described briefly as the map which assigns a connection ∇ to a Riemannian metric g. We analyze the linearization of this map, describing precisely what the kernel is. The finite dimensionality of the kernel of this map is established. As Edgar notes there are two essential conditions for a symmetric connection to be locally metric. One is that the trace of the curvature form has to be zero, and the other one is the existence of a Bianchi tensor. These are, according to Edgar [2], sufficient and necessary conditions for the connection to be locally metric. In order to investigate the global existence of a metric one has to take into account the holonomy group of the connection. The earliest result in this direction was obtained by Schmidt in [4] where he proved the following:

Theorem 1.1 (Schmidt [4]). *Let ∇ be a symmetric connection in* TM. *If the holonomy group of the connection is $\mathcal{H}$ and is a subgroup of the orthogonal group* $\mathrm{O}(n)$, *then ∇ is the Levi-Civita connection of a Riemannian metric.*

Let us investigate the local problem first. Let ∇ denote a symmetric connection on the tangent bundle of a smooth manifold M^n, $(e_i)_{i=1}^n$ a local frame and (θ_i^j) the connection forms of ∇ (i.e. $\nabla e_i = \theta_i^j e_j$). As a consequence of Cartan‘s structural equation

$$\mathrm{d}\theta_i^j = \Omega_i^j - \theta_i^k \wedge \theta_k^j,$$

we see that the trace of the curvature form

$$\mathrm{Tr}(\Omega) = \sum_{i=1}^{n} \Omega_i^i$$

is independent of the choice of the frame and we shall call this two-form the Chern form associated to the connection. The reason is that the cohomology class of this form is independent of the connection and it is also called the first Chern class of the manifold. If the connection is locally a metric connection then the Cartan equation can be rewritten as

$$\Omega_{ij} = \mathrm{d}\theta_{ij} - \theta_{ik} \wedge \theta_{kj},$$

and the curvature matrix in this case being skew symmetric it follows that the trace (which is invariant) has to be zero. Thus, we obtain a necessary condition for a symmetric connection to be locally metrizable. We shall see that this is not sufficient for the local metrizability of a connection. A very restrictive sufficient condition is given by the following lemma:

Lemma 1.1. *Let M be a smooth manifold and ∇ a symmetric flat connection. Then ∇ is locally metrizable.*

The proof of the lemma is elementary and therefore we omit it.