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Björling problem for spacelike, zero mean curvature surfaces in $\mathbb{L}^4$

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Abstract

In this paper, we extend and solve the Björling-type problem for spacelike, zero mean curvature surfaces in the Lorentz–Minkowski four-dimensional space $\mathbb{L}^4$. As an application we establish symmetry principles for this class of surfaces in $\mathbb{L}^4$ and construct new examples.

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1. Introduction

It is well known that spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ represent locally a maximum for the area integral [15,7] and also that they admit a Weierstrass-

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type representation [21,22]. But the spacelike, zero mean curvature surfaces in $\mathbb{L}^4$, represent locally the maximum (resp. minimum) for the area integral, if the normal variation is made in the timelike (resp. spacelike) direction [19]. For these surfaces we also have Weierstrass-type representation [3,12]. An important difference between the global theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ and of the global theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^4$ is established by the so-called *Calabi–Bernstein theorem*. It states that a complete spacelike, zero mean curvature surface in $\mathbb{L}^3$ is a plane [7,8]. However, this result cannot be extended to $\mathbb{L}^n$, $n \geq 4$ [11].

In the three-dimensional Euclidian space $\mathbb{R}^3$, given a real analytic strip (see Section 3), the classical Björling problem [9,16] was proposed by Björling [6] in 1844 and consists of the construction of a minimal surface in $\mathbb{R}^3$ containing the strip in the interior. The solution for this problem was given by Schwarz in [25] by means of a explicit formula in terms of the prescribed strip. This formula gives a beautiful method, besides the Weierstrass representation [24], to construct minimal surfaces with interesting properties. For example, properties of symmetry.

The equivalent problem in the Lorentz–Minkowski three-dimensional space was proposed and solved, using a complex representation formula developed in [1]. The authors introduced the local theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^3$ in a different way of that given in [21,22] through the Weierstrass representation. They constructed new examples of space like, zero mean curvature surfaces, gave alternative proofs of the characterization of the spacelike, zero mean curvature surfaces of revolution and the ruled surfaces in $\mathbb{L}^3$ and proved symmetry principles for those surfaces. We can also find in the work of Gálvez and Mira [13], the version of Björling problem in the hyperbolic three-dimensional space $\mathbb{H}^3$. In that paper the authors constructed the unique mean curvature one surface in $\mathbb{H}^3$ that passes through a given curve with a given unit normal along it, and provide diverse applications.

In Euclidian four-dimensional space, the Björling problem for minimal surfaces was proposed and solved in [4], see also [2], from a complex representation formula. In that work the authors also recovered the symmetry principles of minimal surfaces in $\mathbb{R}^4$ obtained by Eisenhart [10].

In this paper, motivated by results and techniques of [1,4,12], we introduce the local theory of spacelike, zero mean curvature surfaces in $\mathbb{L}^4$, using a complex representation formula – see Theorem 3.1 – that describes the local geometry of these surfaces. This formula is used to solve the Björling problem in $\mathbb{L}^4$, which is illustrated with two examples. As another consequence of Theorem 3.1 we recover the representation formulae of the Björling problem for minimal surfaces in $\mathbb{R}^3$ and spacelike, zero mean curvature surfaces in $\mathbb{L}^3$. We also recover the symmetry principles for these surfaces. Finally, we study the symmetry principles for the spacelike, zero mean curvature surfaces in $\mathbb{L}^4$ and present new examples.

It is not difficult to see that the results in this paper can be extended to spacelike, zero mean curvature surfaces in $\mathbb{L}^n$, $n \geq 4$. Here we restricted the problem to the case $n = 4$ because the formulae and statements are more concise in this case. Also, the case $n = 4$ it is the simplest example of a relativistic spacetime.