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Rotations with unit timelike quaternions in Minkowski 3-space

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Abstract

With the aid of quaternion algebra, rotation in Euclidean space may be dealt with in a simple manner. In this paper, we show that a unit timelike quaternion represents a rotation in the Minkowski 3-space. Also, we express Lorentzian rotation matrix generated with a timelike quaternion.

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1. Introduction

Quaternions were discovered by Sir William R. Hamilton in 1843 and the theory of quaternions was expanded to include applications such as rotations in the early 20th century. The most important property of the quaternions is that every unit quaternion represents a rotation and this plays a special role in the study of rotations in three-dimensional vector spaces. There are various representations for rotations as orthonormal matrices, Euler angles and unit quaternions in the Euclidean space. But to use the unit quaternions is a

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more useful, natural, and elegant way to perceive rotations compared to other methods. A comparison of these methods can be find in [6,7]. Until the middle of the 20th century, the practical use of quaternions has been minimal in comparison with other methods. But, currently, this situation has changed due to progress in robotics, animation and computer graphics technology [6]. Also, quaternions are an efficient way understanding many aspects of physics and kinematics. Today, quaternions are used especially in the area of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations [7].

In this paper, we apply split quaternions to rotations in the Minkowski 3-space. A similar relation to the relationship between quaternions and rotations in the Euclidean space exists between split quaternions and rotations in the Minkowski 3-space. Split quaternions are identified with the semi-Euclidean space $\mathbb{E}_2^4$. Besides, the vector part of split quaternions was identified with the Minkowski 3-space [2]. Thus, it is possible to do with split quaternions many of the things one ordinarily does in vector analysis by using Lorentzian inner and vector products. We give some properties of the split quaternions in Section 3. But, before this, we remind some concepts of quaternions and the Lorentzian space. In the following sections, we demonstrate how timelike split quaternions are used to perform rotations in the Minkowski 3-space.

2. Preliminary

Quaternion algebra $\mathbb{H}$ is an associative, non-commutative division ring with four basic elements $\{1, i, j, k\}$ satisfying the equalities $i^2 = j^2 = k^2 = -1$ and $i * j = k$, $j * k = i$, $k * i = j$, $j * i = -k$, $k * j = -i$, $i * k = -j$ [10]. Quaternions are a generalization of complex numbers. Also, the quaternion algebra is the even subalgebra of the Clifford algebra of the three-dimensional Euclidean space. The Clifford algebra $C\ell(\mathbb{E}_p^n) = C\ell_{n-p,p}$ for the n-dimensional non-degenerate vector space $\mathbb{E}_p^n$ having an orthonormal base $\{e_1, e_2, \ldots, e_n\}$ with the signature $(p, n-p)$ is the associative algebra generated by 1 and $\{e_i\}$ with satisfying the relations $e_i e_j + e_j e_i = 0$ for $\forall i \neq j$ and $e_i^2 = \begin{cases} -1, \text{ if } i = 1, 2, \ldots, p \\ 1, \text{ if } i = p+1, \ldots, n \end{cases}$. The Clifford algebra $C\ell_{n-p,p}$ has the basis $\{e_{i_1} e_{i_2} \ldots e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n\}$. That is, the division algebra of quaternions $\mathbb{H}$ is isomorphic with the even subalgebra $C\ell_{3,0}^+$ of the Clifford algebra $C\ell_{3,0}$ such that $C\ell_{3,0}^+$ has the basis $\{1, e_2 e_3 \to j, e_1 e_3 \to k, e_1 e_2 \to i\}$ [9].

We write any quaternion in the form $q = (q_1, q_2, q_3, q_4) = q_1 + q_2 i + q_3 j + q_4 k$ or $q = Sq + Vq$ where the symbols $Sq = q_1$ and $\vec{V}q = q_2 i + q_3 j + q_4 k$ denote the scalar and vector parts of q. If $Sq = 0$ then q is called pure quaternion. The quaternion product $q * q' = (q_1 + q_2 i + q_3 j + q_4 k) * (q'_1 + q'_2 i + q'_3 j + q'_4 k)$ is obtained by distributing the terms on the right as in ordinary algebra, except that the order of the units must be preserved and then replacing each product of units by the quantity given above.

The conjugate of the quaternion q is denoted by Kq, and defined as $Kq = Sq - \vec{V}q$. The norm of a quaternion $q = (q_1, q_2, q_3, q_4)$ is defined by $\sqrt{q * Kq} = \sqrt{Kq * q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$ and is denoted by Nq and we say that $q_0 = q/Nq$ is unit quater-