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Self-similar solutions to a coagulation equation with multiplicative kernel

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Abstract

Existence of self-similar solutions to the Oort–Hulst–Safronov coagulation equation with multiplicative coagulation kernel is established. These solutions are given by $s(t)^{-\tau}\ \psi_\tau(y/s(t))$ for $(t, y) \in (0, T) \times (0, \infty)$, where T is some arbitrary positive real number, $s(t) = ((3 - \tau)(T - t))^{-1/(3-\tau)}$ and the parameter τ ranges in a given interval $[\tau_c, 3)$. In addition, the second moment of these self-similar solutions blows up at time T. As for the profile ψ_τ, it belongs to $L^1(0, \infty; y^2 \mathrm{d}y)$ for each $\tau \in [\tau_c, 3)$ but its behaviour for small and large y varies with the parameter τ.

Keywords: Coagulation; Self-similar solution; Gelation; Blow-up; Multiplicative kernel

1. Introduction

Coagulation equations are mean-field models describing the dynamics of the mass distribution function of a system of particles growing by successive mergers. This class of models includes in particular the well-known Smoluchowski coagulation equation [16,17]

$$\partial_t f(t, y) = \frac{1}{2} \int_0^y a(y_*, y - y_*)\ f(t, y - y_*)\ f(t, y_*)\, \mathrm{d}y_* - f(t, y) \int_0^\infty a(y, y_*)\ f(t, y_*)\, \mathrm{d}y_*, \qquad (1)$$

where $f(t, y) \geq 0$ denotes the mass distribution function of particles with mass $y \in (0, \infty)$ at time $t \geq 0$. The coagulation kernel $a(y, y_*)$ describes the likelihood that the encounter of a particle of mass y with a particle of mass y_* produces a particle of mass $y + y_*$ and satisfies $a(y, y_*) = a(y_*, y) \geq 0$. Observing that mass is obviously conserved during each coagulation event, a reasonable expectation is that the total mass

$$M_1(t) := \int_0^\infty y\ f(t, y)\, \mathrm{d}y$$

of the system of particles at time t should remain constant throughout time evolution. It is however well known now that this property fails to be true for coagulation kernels growing sufficiently rapidly for large y, y_*, such as $a(y, y_*) = (y\ y_*)^{\lambda/2}$ for $\lambda \in (1, 2]$. For such kernels, there is actually a runaway growth which produces particles with infinite mass in a finite time, a phenomenon called the occurrence of gelation (see, e.g., the review articles [1,3,8,10] for more information). Let us mention at this point that, though formal arguments predicting the occurrence of gelation have been known for some time, a rigorous proof has only been supplied recently in [6] by probabilistic arguments and in [5] by deterministic arguments.

Now, introducing the gelation time

$$T_{\mathrm{gel}} := \inf\{t \geq 0 \text{ such that } M_1(t) < M_1(0)\}$$

and assuming that T_{gel} is finite, a detailed analysis of the behaviour of $f(t)$ just before the gelation time is required to elucidate the gelation mechanism. For homogeneous coagulation kernels such as $a(y, y_*) = (y\ y_*)^{\lambda/2}$ for $\lambda \in (1, 2]$, it is commonly believed that such a behaviour is self-similar, that is, there are $\tau \in (0, \infty)$, $s : [0, T_{\mathrm{gel}}) \to (0, \infty)$ and $\varphi : (0, \infty) \to (0, \infty)$ such that

$$s(t) \to \infty \quad \text{and} \quad f(t, y) \sim f_S(t, y) := \frac{1}{s(t)^\tau}\ \varphi\left(\frac{y}{s(t)}\right) \qquad (2)$$
$$\text{as } t \to T_{\mathrm{gel}}.$$

A first task is then to look for self-similar solutions f_S to (1) as described in (2). This problem seems however to be

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of considerable difficulty and is only completely solved for the multiplicative coagulation kernel $a(y, y_*) = y\ y_*$ [3,12]. For other kernels, no result is available to our knowledge. Actually, the first difficulty to be faced is determining the value of the exponent τ. For homogeneous coagulation kernels with homogeneity degree $\lambda \in (1, 2]$ (i.e. $a(\xi y, \xi y_*) = \xi^\lambda\ a(y, y_*)$ for $(\xi, y, y_*) \in (0, \infty)^3$), the value $\tau = (\lambda + 3)/2$ has been proposed in [2] but numerical simulations performed in [9,11] seem to indicate a different value of τ. We refer the reader to the review article [10] for a thorough discussion of this issue. In fact, for the only case which is solved, namely $a(y, y_*) = y\ y_*$, there is an interval of values of τ for which there exists a self-similar solution $f_\tau(t, y) := s(t)^{-\tau}\ \varphi_\tau(y/s(t))$ to (1) [12]. More precisely, it follows from [12] that, if $a(y, y_*) = y\ y_*$, $\tau \in [5/2, 3)$ and $T > 0$, there is a self-similar solution $f_\tau(t, y) := s(t)^{-\tau}\ \varphi_\tau(y/s(t))$ to (1) with $s(t) = (T - t)^{-1/(3-\tau)}$,

$$\varphi_{5/2}(y) := (4\pi)^{-1/2}\ y^{-5/2}\ e^{-y/4}$$

and, for $\tau \in (5/2, 3)$,

$$\varphi_\tau(y) \sim c_0\ y^{-\tau} \quad \text{as } y \to 0 \quad \text{and}$$
$$\varphi_\tau(y) \sim c_\infty\ y^{-(2\tau-3)/(\tau-2)} \quad \text{as } y \to \infty$$

for some positive constants c_0 and c_∞ depending only on τ. For that particular case, the homogeneity degree of a is $\lambda = 2$ and the value $5/2 = (\lambda + 3)/2$ of τ suggested in [2] does indeed lie within the range of values of τ for which a self-similar solution does exist. It is nevertheless a peculiar value since it is the only value of τ for which φ_τ has a finite third moment (and actually decays exponentially fast as $y \to \infty$). As a final comment, let us mention that the Smoluchowski coagulation equation (1) with the multiplicative kernel $a(y, y_*) = y\ y_*$ can be reduced to a simpler problem by applying a Laplace transform. Thanks to this property, the existence of self-similar solutions can be proved, and the question of convergence studied as well [12]. However, this trick does not work for other gelling kernels.

Besides the Smoluchowski coagulation equation (1), there are other coagulation equations to which the previous discussion on the gelation phenomenon equally applies, and in particular the Oort–Hulst–Safronov (OHS) coagulation equation [14,15]

$$\partial_t f(t, y) = -\partial_y \left(f(t, y) \int_0^y y_*\ a(y, y_*)\ f(t, y_*)\ dy_* \right) - f(t, y) \int_y^\infty a(y, y_*)\ f(t, y_*)\ dy_*. \tag{3}$$

For the OHS equation (3), the occurrence of gelation is also known to take place for $a(y, y_*) = (y\ y_*)^{\lambda/2}$ with $\lambda \in (1, 2]$ [4,7]. The purpose of this work is then to show that, still for the multiplicative kernel $a(y, y_*) = y\ y_*$, a family $f_\tau(t, y) := s(t)^{-\tau}\ \psi_\tau(y/s(t))$ of self-similar solutions to (3) can be constructed, the parameter τ ranging in a non-empty interval $[\tau_c, 3)$ with $\tau_c < 5/2$. Our result shows in particular that, in that case, the value $5/2$ does not seem to play any special role.

From now on, we thus assume that $a(y, y_*) = y\ y_*$ for $(y, y_*) \in (0, \infty)^2$ and look for self-similar solutions f_S to (3) of the form

$$f_S(t, y) = \frac{1}{s(t)^\tau}\ \psi\left(\frac{y}{s(t)}\right), \quad (t, y) \in (0, \infty)^2, \tag{4}$$

the parameter τ and the functions s and ψ to be determined with the requirement that $s(t) \to \infty$ as $t \to T$ for some fixed $T > 0$. Inserting the self-similar ansatz (4) in (3) yields the existence of a real number w such that

$$\frac{ds}{dt}(t)\ s(t)^{\tau-4} = w, \tag{5}$$

$$w \left(\tau\ \psi(y) + y\ \frac{d\psi}{dy}\right) = \frac{d}{dy}\left(y\ \psi(y) \int_0^y y_*^2\ \psi(y_*)\ dy_*\right) + y\ \psi(y) \int_y^\infty y_*\ \psi(y_*)\ dy_* \tag{6}$$

for $(t, y) \in (0, T) \times (0, \infty)^2$. Since we expect the function s to be an increasing function of time which blows up at time T, these two properties imply that

$$w > 0 \quad \text{and} \quad \tau < 3. \tag{7}$$

Consequently, $s(t) = (w(3-\tau)(T-t))^{-1/(3-\tau)}$ for $t \in [0, T)$. We next observe that, if ψ satisfies (6), then so does $\psi_{\varrho,\sigma}(y) := \varrho\ \psi(\sigma y)$ with $\varrho\ w\ \sigma^{-3}$ instead of w. We may thus eliminate the parameter w and fix it to the value $w = 1$. Then,

$$s(t) = ((3 - \tau)(T - t))^{-1/(3-\tau)}, \quad t \in [0, T). \tag{8}$$

Having identified the function s in terms of τ, it remains to figure out for which values of τ the Eq. (6) with $w = 1$ has a meaningful solution. Besides non-negativity, we will require that ψ has a finite second moment, that is,

$$\psi \in L^1(0, \infty; y^2 dy), \quad \psi \ge 0 \quad \text{a.e. in } (0, \infty),\ \psi \not\equiv 0, \tag{9}$$

which is somehow a minimal requirement on ψ for f_S to solve (3) at least in a weak sense. Furthermore, if ψ fulfils (9), we deduce from (4) that

$$\int_0^\infty y^2\ f_S(t, y)\ dy = \frac{1}{(3 - \tau)(T - t)} \int_0^\infty y^2\ \psi(y)\ dy,$$

so that the second moment of f_S blows up at time T. In this connection, we recall that the occurrence of gelation of a solution f to (3) with $a(y, y_*) = y\ y_*$ is related to the blow-up of the second moment of f [4]. Indeed, it is conjectured that the gelation time and the blow-up time of the second moment of f coincide.

Our main result is then to exhibit a range of values of τ for which there exists a solution to (6) with $w = 1$ satisfying (9). In order to state it, some notation and preliminary results are needed which we gather below. We denote by τ_c ($\tau_c \sim 2.255$) the unique real number in $(2, 3)$ such that

$$(\tau_c - 1)\ \ln\left(\frac{\tau_c - 1}{\tau_c - 2}\right) = 2. \tag{10}$$

For $\tau \in [\tau_c, 3)$, the function g_τ defined by

$$g_\tau(z) := 2z - (\tau - 1)\ \ln\left(1 + \frac{z}{\tau - 2}\right), \quad z \in [0, \infty), \tag{11}$$