STRUCTURE OF THE PLANAR GALILEAN CONFORMAL ALGEBRA

SHOULAN GAO

Department of Mathematics, Huzhou Teachers College,
Huzhou 313000, China
(e-mail: gaoshoulan@hutc.zj.cn)

DONG LIU

Department of Mathematics, Huzhou Teachers College,
Huzhou 313000, China
(e-mail: liudong@hutc.zj.cn)

and

YUFENG PEI[1]

Department of Mathematics, Shanghai Normal University,
Shanghai 200234, China
(e-mail: pei@shnu.edu.cn)

(*Received October 13, 2015 – Revised December 21, 2015*)

In this paper, we compute the low-dimensional cohomology groups of the planar Galilean conformal algebra introduced by Bagchi and Goparkumar. Consequently we determine its derivations, central extensions, and automorphisms.

Keywords: planar Galilean conformal algebra, cohomology, automorphism.

1. Introduction

The Galilean conformal algebras (GCAs) have recently been studied in the context of the nonrelativistic limit of the AdS/CFT conjecture correspondence [7]. The finite-dimensional GCA is associated with certain non-semisimple Lie algebra which is regarded as a nonrelativistic analogue of conformal algebras. It was found that the finite GCA could be given an infinite-dimensional lift for all space-time dimensions (see [7, 8, 15, 24]). These infinite-dimensional extensions have the (centerless) Virasoro algebra as a subalgebra, which would suggest they are of physical importance. In particular, the infinite-dimensional GCA in 2D turned out to be related to the symmetries of nonrelativistic hydrodynamic equations [1], the BMS/GCA correspondence [9], the tensionless limit of string theory [10] and statistical mechanics [14]. It is worth pointing out that these algebras have been

[1] Y. Pei, corresponding author.

widely studied in the context of Lie algebras and vertex operator algebras (see [16, 20, 31]).

In this paper, we focus on the planar Galilean conformal algebra $\mathfrak{g}$, which was first introduced by Bagchi and Gopakumar in [7] and named by Aizawa and Kimura in [1]. Recall that $\mathfrak{g}$ is a Lie algebra with the basis $\{L_m, H_m, I_m, J_m \mid m \in \mathbb{Z}\}$ and the nontrivial Lie brackets are defined by

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n}, & [L_m, H_n] &= -nH_{m+n}, \\ [L_m, I_n] &= (m-n)I_{m+n}, & [L_m, J_n] &= (m-n)J_{m+n}, \\ [H_m, I_n] &= J_{m+n}, & [H_m, J_n] &= -I_{m+n}, \end{aligned}$$

for $m, n \in \mathbb{Z}$. It has a natural realization in terms of differential operators over the Laurent polynomial ring $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, t^{\pm 1}]$ (see [7]):

$$\begin{aligned} &L_n = -t^{n+1}\partial_t - (n+1)t^n(x_1\partial_{x_1} + x_2\partial_{x_2}), \\ &H_n = -t^n(x_1\partial_{x_2} - x_2\partial_{x_1}), \qquad I_n = -t^n\partial_{x_1}, \qquad J_n = -t^n\partial_{x_2}. \end{aligned}$$

for $n \in \mathbb{Z}$. Some representations of certain central extension of $\mathfrak{g}$ have been investigated in [1, 2, 4] Moreover, the subalgebra $\mathfrak{g}_f$ generated by $\{L_m, H_0, I_m, J_m \mid m = 0, \pm 1\}$ is isomorphic to the finite-dimensional GCA, which has an exotic central extension $\hat{\mathfrak{g}}_f$ (see [23]). The Verma modules and simple modules of $\hat{\mathfrak{g}}_f$ have been studied in [3, 22].

Cohomology groups are important in the structure theory of Lie algebras. Their determination frequently provides insight into invariants of Lie algebras (see [13, 29]). In general, however, direct computation of these groups (even in low-dimensional case) is a nontrivial problem. In this paper some general results concerning low-dimensional cohomology groups and automorphisms of semi-direct product Lie algebras are established. These are applied in order to determine a classification of outer derivations, central extensions and automorphisms of the planar Galilean conformal algebra $\mathfrak{g}$.

This paper is organized as follows: In Section 2, by the low degree terms of Hochschild–Serre spectral sequence, we compute the first cohomology groups with coefficients in the adjoint module. It follows that we obtain all derivations with coefficients in the adjoint module. The second cohomology group with trivial coefficients of $\mathfrak{g}$ was computed earlier by Aizawa and Kimura in [1]. In Section 3, we give an alternative and easier proof by using a decomposition of the 2nd cohomology group for general semi-product Lie algebras. It follows that we determine the universal central extension of $\mathfrak{g}$. In Section 3, we describe explicitly the structure of the group of automorphisms of $\mathfrak{g}$ in terms of a decomposition of automorphism groups for general semi-product Lie algebras.

We denote by $\mathbb{Z}$ the set of all integers, $\mathbb{Z}^*$ the set of all nonzero integers, $\mathbb{C}$ the set of all complex numbers, and $\mathbb{C}^*$ the set of all nonzero complex numbers. Throughout this paper all algebras are over the complex number field $\mathbb{C}$.