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Nonlinear wave dynamics near phase transition in $\mathcal{PT}$-symmetric localized potentials

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HIGHLIGHTS

- Analysis for the NLS equation with a $\mathcal{PT}$-symmetric potential is presented.
- A $\mathcal{PT}$-Krein theory is developed for phase transition in this $\mathcal{PT}$-symmetric system.
- Nonlinear dynamics near phase transition is derived analytically.
- Comparison with numerics shows good agreement.

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ABSTRACT

Nonlinear wave propagation in parity-time symmetric localized potentials is investigated analytically near a phase-transition point where a pair of real eigenvalues of the potential coalesce and bifurcate into the complex plane. Necessary conditions for a phase transition to occur are derived based on a generalization of the Krein signature. Using the multi-scale perturbation analysis, a reduced nonlinear ordinary differential equation (ODE) is derived for the amplitude of localized solutions near phase transition. Above the phase transition, this ODE predicts a family of stable solitons not bifurcating from linear (infinitesimal) modes under a certain sign of nonlinearity. In addition, it predicts periodically-oscillating nonlinear modes away from solitons. Under the opposite sign of nonlinearity, it predicts unbounded growth of solutions. Below the phase transition, solution dynamics is predicted as well. All analytical results are compared to direct computations of the full system and good agreement is observed.

1. Introduction

Parity-time ($\mathcal{PT}$) symmetric systems started out from an observation in non-Hermitian quantum mechanics, where a complex but $\mathcal{PT}$-symmetric potential could possess all-real spectrum [1]. This concept later spread out to optics, Bose–Einstein condensation, mechanical systems, electric circuits and many other fields, where a judicious balancing of gain and loss constitutes a $\mathcal{PT}$-symmetric system which can admit all-real linear spectrum [2–16]. For example, in optics, an even refractive index profile together with an odd gain–loss landscape yields a $\mathcal{PT}$-symmetric system. A common phenomenon in linear $\mathcal{PT}$-symmetric systems is the existence of a phase transition (also known as $\mathcal{PT}$-symmetry breaking), where pairs of real eigenvalues collide and then bifurcate to the complex plane when the magnitude of gain and loss is above a certain threshold [1,7,17–19]. This phase transition has been observed experimentally in a wide range of physical systems [4–6,8,13,14]. When nonlinearity is present in $\mathcal{PT}$ systems, the interplay between nonlinearity and $\mathcal{PT}$ symmetry gives rise to additional novel properties such as the existence of continuous families of stationary nonlinear modes, stabilization of nonlinear modes above phase transition, and symmetry breaking of nonlinear modes [7,18–28]. These findings reveal that $\mathcal{PT}$-symmetric systems break the boundary between traditional conservative and dissipative systems and open new exciting research territories. Practical applications of $\mathcal{PT}$ systems are emerging as well, such as recent demonstrations of $\mathcal{PT}$-symmetric micro-ring lasers and unidirectional reflectionless $\mathcal{PT}$ metamaterials [12,15,16].

Phase transition is an important property of linear $\mathcal{PT}$-symmetric systems which is at the heart of many proposed applications [12,16,29]. At a phase transition, a pair of real eigenvalues coalesce and form an exceptional point featuring a non-diagonal Jordan block (i.e., the algebraic multiplicity of the eigenvalue is higher than the geometric multiplicity). In the presence of nonlinearity (such as when the wave amplitude is not small), the interplay between the phase transition and nonlinearity is a fascinating

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subject. This interplay was previously studied for periodic $\mathcal{PT}$-symmetric potentials in [25,30,31], where novel behaviors such as wave-blowup and temporally-oscillating bound states were reported below phase transition. In addition, stable nonlinear Bloch modes were reported above phase transition because nonlinearity transforms the effective potential from above to below phase transition [25] (a similar phenomenon was reported in [32] for a different $\mathcal{PT}$-symmetric dimer model). However, in periodic potentials above phase transition, the presence of unstable infinitely extended linear modes makes the zero background unstable everywhere, which excludes the possibility of stable spatially-localized coherent structures. In localized potentials, will the situation be different?

In this article we study nonlinear wave behaviors in localized $\mathcal{PT}$-symmetric potentials near a phase transition. Unlike periodic potentials, the instability of linear modes above phase transition is limited to the area around the localized potential. In this case, the addition of nonlinearity can balance against the effects of gain and loss making stable spatially-localized coherent structures, such as solitons and oscillating bound states, possible above phase transition. Mathematically, we explain this phenomenon by a multiscale perturbation analysis, where a reduced nonlinear ordinary differential equation (ODE) is derived for the amplitude of localized solutions near phase transition. Above phase transition, this ODE model predicts a family of stable solitons not bifurcating from linear (infinitesimal) modes under a certain sign of nonlinearity. In addition, it predicts persistent oscillating nonlinear modes away from solitons. Under the opposite sign of nonlinearity, it predicts unbounded growth of solutions. Similarly, solution dynamics below phase transition is predicted as well. All these predictions are verified in the full partial differential equation (PDE) system. In addition to these nonlinear dynamics, we also derive a necessary condition for a phase transition to occur at an exceptional point in the linear $\mathcal{PT}$ system by a generalization of the Krein signature, namely, a phase transition from a collision of two real eigenvalues is possible only when the two eigenvalues have opposite $\mathcal{PT}$-Krein signatures.

2. Preliminaries

The mathematical model we consider in this article is the following potential NLS equation

$$i\psi_z + \psi_{xx} + V(x;\epsilon)\psi + \sigma|\psi|^2\psi = 0, \tag{2.1}$$

where $V(x;\epsilon)$ is a $\mathcal{PT}$-symmetric complex potential, i.e.,

$$V^*(-x;\epsilon) = V(x;\epsilon), \tag{2.2}$$

parameterized by ϵ, $\sigma = \pm 1$ is the sign of nonlinearity, and the superscript * represents complex conjugation. Throughout the text, we assume that the potential $V(x;\epsilon)$ is continuous with ϵ. Eq. (2.1) governs nonlinear light propagation in an optical medium with gain and loss [18] as well as the dynamics of Bose–Einstein condensates in a double-well potential with atoms injected into one well and removed from the other well [9,10]. $\mathcal{PT}$-symmetric optical systems have been realized experimentally [5,6,12,14–16], however $\mathcal{PT}$-symmetric Bose–Einstein condensates remain theoretical. Without loss of generality, we assume a phase transition occurs at $\epsilon = 0$, where a pair of real eigenvalues of the potential coalesce and form an exceptional point, whose algebraic multiplicity is two and the geometric multiplicity is one. We will analyze the solution dynamics in Eq. (2.1) near this exceptional point, i.e., when $|\epsilon| \ll 1$.

The analysis to be developed applies to all localized $\mathcal{PT}$-symmetric potentials near a phase transition. To illustrate these analytical results and compare them with direct numerics of the full model (2.1), we will use a concrete example—the so-called Scarff II potential

$$V = V_R\,\mathrm{sech}^2(x) + iW_0\,\mathrm{sech}(x)\tanh(x), \tag{2.3}$$

where V_R and W_0 are real parameters. For this potential, phase transition occurs at $W_0 = V_R + 1/4$ [17], and solitons as well as robust oscillating solutions have been reported numerically below phase transition in [18,33–35].

3. $\mathcal{PT}$-Krein signature and a necessary condition for phase transition

For the potential NLS equation (2.1), when one looks for linear eigenmodes $\psi = u(x)e^{-i\mu z}$, with regards to the stability of the zero state, the eigenvalue problem

$$L(x;\epsilon)u = -\mu u \tag{3.1}$$

will be obtained, where

$$L = \partial_{xx} + V(x;\epsilon) \tag{3.2}$$

is a Schrödinger operator with a complex $\mathcal{PT}$-symmetric potential, and μ is an eigenvalue. We wish to consider the phase-transition process by which the spectrum of L changes from all-real to partially-complex. This phase transition occurs when a pair of real eigenvalues collide, forming an exceptional point, and then bifurcate into the complex plane. It is important to recognize that not any two real eigenvalues can turn complex upon collision. This is analogous to the linear stability of equilibria in Hamiltonian systems, where not just any two purely imaginary eigenvalues upon collision can bifurcate off the imaginary axis and result in linear instability [36–39]. With this in mind, we consider the question: under what conditions can a pair of real eigenvalues of L induce a phase transition upon collision?

We will work in the square-integrable doubly-differentiable Hilbert functional space H^2 endowed with the standard inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)\,dx.$$

Under this inner product, the adjoint operator $L^\dagger$ of L is

$$L^\dagger = L^* = \partial_{xx} + V^*(x;\epsilon).$$

When the potential $V(x)$ is $\mathcal{PT}$-symmetric, a key property of the operator L, which can be readily verified, is

$$L^\dagger = \mathcal{P}L\mathcal{P}^{-1}, \tag{3.3}$$

where $\mathcal{P}$ is the parity operator, i.e., $\mathcal{P}f(x) \equiv f(-x)$. For this parity operator, $\mathcal{P}^{-1} = \mathcal{P}$ and $\mathcal{P}^\dagger = \mathcal{P}$, thus $\mathcal{P}$ is Hermitian and invertible. Consequently, L is pseudo-Hermitian [40] and $\mathcal{P}L$ is Hermitian.

One of the consequences of the pseudo-Hermiticity of L is that, any complex eigenvalues of L must come in conjugate pairs (μ,μ^*). The reason is that under pseudo-Hermiticity, $L^\dagger$ is similar to L, thus $L^\dagger$ and L share the same spectrum. But the spectrum of $L^\dagger$ is the complex conjugate of L's, thus complex eigenvalues of L must come in (μ,μ^*) pairs.

Another consequence of the pseudo-Hermiticity of L is that, it allows us to define a $\mathcal{PT}$-Krein signature for discrete real eigenvalues of L, which will prove to be important when studying phase transition from collisions of L's real eigenvalues. For this purpose, we endow the Hilbert space H^2 with another indefinite $\mathcal{PT}$ inner product [41]

$$\langle f, g\rangle_{\mathcal{PT}} \equiv \langle f, \mathcal{P}g\rangle = \int_{-\infty}^{\infty} f^*(x)g(-x)\,dx. \tag{3.4}$$