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Stability and transitions of the second grade Poiseuille flow

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HIGHLIGHTS

- Pipe Poiseuille flow of a second grade fluid is considered.
- Energy and linear stability thresholds for the Reynolds number are found.
- Possible transition scenarios are continuous or catastrophic.
- For small second order viscous effects, the preferred transition is catastrophic.

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ABSTRACT

In this study we consider the stability and transitions for the Poiseuille flow of a second grade fluid which is a model for non-Newtonian fluids. We restrict our attention to perturbation flows in an infinite pipe with circular cross section that are independent of the axial coordinate.

We show that unlike the Newtonian ($\epsilon = 0$) case, in the second grade model ($\epsilon > 0$ case), the time independent base flow exhibits transitions as the Reynolds number R exceeds the critical threshold $R_c = 8.505\epsilon^{-1/2}$ where ϵ is a material constant measuring the relative strength of second order viscous effects compared to inertial effects.

At $R = R_c$, we find that the transition is either continuous or catastrophic and a small amplitude, time periodic flow with 3-fold azimuthal symmetry bifurcates. The time period of the bifurcated solution tends to infinity as R tends to R_c. Our numerical calculations suggest that for low ϵ values, the system prefers a catastrophic transition where the bifurcation is subcritical.

We also show that there is a Reynolds number R_E with $R_E < R_c$ such that for $R < R_E$, the base flow is globally stable and attracts any initial disturbance with at least exponential speed. We show that the gap between R_E and R_c vanishes quickly as ϵ increases.

1. Introduction

Certain natural materials manifest some fluid characteristics that cannot be represented by well-known linear viscous fluid models. Such fluids are generally called non-Newtonian fluids. There are several models that have been proposed to predict the non-Newtonian behavior of various type of materials. One class of fluids which has gained considerable attention in recent years is the fluids of grade n [1–7]. A great deal of information of these types of fluids can be found in [8]. Among these fluids, one special subclass associated with second order truncations is the so called second-grade fluids. The constitutive equation of a second grade fluid is given by the following relation for incompressible fluids

$$\mathbf{t} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,$$

where $\mathbf{t}$ is the stress tensor, p is the pressure, μ is the classical viscosity, α_1 and α_2 are the material coefficients. $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors defined by

$$\mathbf{A}_1 = \nabla\mathbf{v} + \nabla\mathbf{v}^T,$$
$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1\nabla\mathbf{v} + \nabla\mathbf{v}^T\mathbf{A}_1,$$

where $\mathbf{v}$ is the velocity field and the overdot represents the material derivative with respect to time. This type of constitutive relation was first proposed in [9]. The conditions

$$\alpha_1 + \alpha_2 = 0, \qquad \mu \geq 0, \quad \alpha_1 \geq 0,$$

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must be satisfied for the second-grade fluid to be entirely consistent with classical thermodynamics and the free energy function achieves its minimum in equilibrium [10].

Equation of motion for an incompressible second grade Rivlin–Ericksen fluid is represented as

$$\rho\left(\mathbf{v}_t+\mathbf{w}\times\mathbf{v}+\nabla\frac{|\mathbf{v}|^2}{2}\right) = -\nabla p+\mu\Delta\mathbf{v}+\alpha\left[\Delta\mathbf{v}_t+\Delta\mathbf{w}\times\mathbf{v}+\nabla\left(\mathbf{v}\cdot\Delta\mathbf{v}+\frac{1}{4}|\mathbf{A}_1|^2\right)\right],$$

$$\nabla\cdot\mathbf{v}=0$$

where ρ is the density, $\alpha=\alpha_1=-\alpha_2$, represents the second order material constant. Subscript t denotes the partial derivative with respect to time, $\mathbf{w}$ is the usual vorticity vector defined by

$$\mathbf{w}=\nabla\times\mathbf{v}.$$

We next define the non-dimensional variables

$$\mathbf{v}^*=\frac{\mathbf{v}}{U},\qquad p^*=\frac{p}{\rho U^2},\qquad t^*=\frac{tU}{L},\qquad \mathbf{x}^*=\frac{\mathbf{x}}{L},$$

where U and L are characteristic velocity and length, respectively. By letting ϵ represent the second order non-dimensional material constant which measures the relative strength of second order viscous effects compared to inertial effects and defining the Reynolds number,

$$\mathrm{R}=\frac{\rho UL}{\mu},\qquad \epsilon=\frac{\alpha}{\rho L^2},$$

the equation of motion, with asterisks omitted, can be expressed as

$$\nabla\bar{p}=\frac{1}{\mathrm{R}}\Delta\mathbf{v}+\epsilon(\Delta\mathbf{w}\times\mathbf{v}+\Delta\mathbf{v}_t)-\mathbf{v}_t-\mathbf{w}\times\mathbf{v}, \tag{1}$$

where the characteristic pressure $\bar{p}$ is defined as

$$\bar{p}=p+\frac{|\mathbf{v}|^2}{2}-\epsilon\left(\mathbf{v}\Delta\mathbf{v}+\frac{1}{4}|\mathbf{A}_1|^2\right).$$

Taking curl of both sides of (1) we can simply write the equation of motion as

$$\nabla\times\left[\frac{1}{\mathrm{R}}\Delta\mathbf{v}+\epsilon(\Delta\mathbf{w}\times\mathbf{v}+\Delta\mathbf{v}_t)-\mathbf{v}_t-\mathbf{w}\times\mathbf{v}\right]=0, \tag{2}$$

which is the field equation of incompressible unsteady second grade Rivlin–Ericksen fluid independent of the choice of any particular coordinate system.

Now we restrict our interest to flows in an infinite cylinder with circular cross section and consider the no-slip boundary conditions. If we choose the characteristic length L to be the radius of the tube, then Eqs. (2) admit the following steady state solution, known as the pipe Poiseuille flow,

$$\mathbf{v}_0=\left(0,0,\frac{\overline{P_0}\mathrm{R}}{4}(1-x^2-y^2)\right),$$

corresponding to a steady pressure p_0. Here the constant $\overline{P_0}=-\frac{\partial p_0}{\partial z}$ is the nondimensional axial pressure gradient. By recalling the definition of nondimensional pressure, we observe that the dimensional axial pressure gradient is given by $P_0=\frac{\rho U^2\overline{P_0}}{L}$. Now if we choose the characteristic velocity as $U=\frac{P_0L^2}{4\mu}$, the steady state solution becomes

$$\mathbf{v}_0=(0,0,1-x^2-y^2).$$

Now we consider perturbations $\mathbf{v}'(x,y,t)=\mathbf{v}-\mathbf{v}_0$ from this basic flow which depend only on the two cross-sectional variables x, y and the time t. Introducing the polar coordinates $\mathbf{v}''(t,r,\theta)=\mathbf{v}'(t,r\cos\theta,r\sin\theta)$, introducing a stream function ψ such that $\mathbf{v}=(\psi_y,-\psi_x,w)$ and ignoring the primes, the equations become

$$\begin{aligned}\frac{\partial}{\partial t}(1-\epsilon\Delta)w &= \frac{1}{\mathrm{R}}\Delta w+2\psi_\theta+J(\psi,(1-\epsilon\Delta)w),\\ \frac{\partial}{\partial t}\Delta(\epsilon\Delta-1)\psi &= -\frac{1}{\mathrm{R}}\Delta^2\psi+2\epsilon\Delta w_\theta+J((1-\epsilon\Delta)\Delta\psi,\psi)\\ &\quad+\epsilon J(\Delta w,w),\end{aligned} \tag{3}$$

in the interior of the unit disk Ω where J is the advection operator

$$J(f,g)=\frac{1}{r}(f_rg_\theta-f_\theta g_r).$$

The field equations are supplemented with no-slip boundary conditions for the velocity field

$$w=\psi=\frac{\partial\psi}{\partial r}=0\quad\text{at } r=1. \tag{4}$$

In this paper, our main aim is to investigate the stability and transitions of (3) subject to (4). We first prove that unlike the Newtonian case ($\epsilon=0$), in the non-Newtonian case ($\epsilon>0$), the system undergoes a dynamic transition at the critical Reynolds number $\mathrm{R}_c=8.505\epsilon^{-1/2}$. As R crosses R_c the steady flow loses its stability, and a transition occurs. If we denote the azimuthal wavenumber of an eigenmode by m, then two modes, called critical modes hereafter, with $m=3$ and radial wavenumber 1, become critical at $\mathrm{R}=\mathrm{R}_c$.

Recently, the dynamic transition theory has gained much attention. We refer the readers to [11] for the details of this theory, and [12–19] for several recent applications. Using the language of dynamic transition theory, we can show that the transition at $\mathrm{R}=\mathrm{R}_c$ is either Type-I (continuous) or Type-II (catastrophic). In Type-I transitions, the amplitudes of the transition states stay close to the base flow after the transition. Type-II transitions, on the other hand, are associated with more complex dynamical behavior, leading to metastable states and local attractors far away from the base flow.

We show that the type of transition at $\mathrm{R}=\mathrm{R}_c$ preferred for this problem is determined by a complex parameter A given by (28) which only depends on ϵ. In the generic case of nonzero real part of A, there are two possible transition scenarios depending on the sign of the real part of A: Type-I (continuous) transition if $\mathrm{Re}(A)<0$ and Type-II (catastrophic) transition if $\mathrm{Re}(A)>0$.

In the continuous transition scenario ($\mathrm{Re}(A)<0$ case), a stable, small amplitude, time periodic flow with 3-fold azimuthal symmetry bifurcates on $\mathrm{R}>\mathrm{R}_c$. The time period of the bifurcated solution tends to infinity as R tends to R_c, a phenomenon known as infinite period bifurcation [20]. The dual scenario is the catastrophic transition ($\mathrm{Re}(A)>0$ case) where the bifurcation is subcritical on $\mathrm{R}<\mathrm{R}_c$ and a repeller bifurcates.

The transition number A depends on the system parameter ϵ in a non-trivial way, and it is not possible to find an explicit expression of A as a function of ϵ. So, A must be computed numerically for a given ϵ. Physically, the transition number A can be considered as a measure of net mechanical energy transferred from all modes back to the critical modes which in turn modify the base flow. We show that A is determined by the nonlinear interactions of the critical modes ($m=3$) with all the modes having azimuthal wavenumber $m=0$ and $m=6$. Moreover, our numerical computations suggest that for low ϵ fluids ($\epsilon<1$), just a single nonlinear interaction, namely the one with $m=0$ and radial wavenumber 1 mode, dominates all the rest contributions to A. Our numerical experiments with low ϵ, i.e. $\epsilon<1$, suggest that the real part of A is always positive indicating a catastrophic transition on $\mathrm{R}>\mathrm{R}_c$.

We also determine the Reynolds number threshold $\mathrm{R}_E>0$ below which the Poiseuille flow is globally stable, attracting all