REDUCTION FORMULAE FOR SYMMETRIC PRODUCTS OF SPIN MATRICES

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(*Received July 31, 2015 – Revised September 14, 2015*)

We show that, for SU(2) generators of arbitrary dimension D there exist identities that express the completely symmetric product of D matrices in terms of completely symmetric products of fewer number of matrices. We also indicate why such identities are important in characterizing electromagnetic interactions of particles.

Keywords: SU(2) representations, spin, electromagnetic interactions.
PACS numbers: 02.20.Sv.

1. Introduction

The purpose of this article is to present several identities involving spin matrices, or equivalently SU(2) generators [1]. The identities are independent of any basis used for writing the explicit forms of the generators, and can be used to write products of a certain number of spin matrices by using products of smaller number of such matrices.

The SU(2) generators, or the spin matrices, satisfy the commutation relation [2]

$$\left[S_i, S_j\right] = i \sum_k \varepsilon_{ijk} S_k \tag{1.1}$$

where the summation [3] over k runs from 1 to 3 since there are three generators, and ε_{ijk} is the completely antisymmetric symbol with

$$\varepsilon_{123} = +1. \tag{1.2}$$

Eq. (1.1) shows that an antisymmetric product of two spin matrices yields terms with only one spin matrix. From this, it is easy to show that if any expression containing spin matrices is antisymmetric in two indices, the expression can be written by using smaller number of spin matrices. For example, consider the string $S_i S_j S_k - S_k S_j S_i$, which is antisymmetric under the interchange of the indices i and k. Notice that

$$S_i S_j S_k - S_k S_j S_i = \big[S_i, S_j\big] S_k + S_j \big[S_i, S_k\big] + \big[S_j, S_k\big] S_i$$
$$= i \sum_l \left(\varepsilon_{ijl} S_l S_k + \varepsilon_{ikl} S_j S_l + \varepsilon_{jkl} S_l S_i\right), \tag{1.3}$$

using Eq. (1.1) to write the last step. This example shows that the product of three spin matrices, antisymmetrized in two indices, can be expressed as a sum of terms which contain products of only two spin matrices. A trivial generalization of Eq. (1.3) is the following:

$$A S_i B S_k C - A S_k B S_i C = A\big[S_i, B\big] S_k C + AB\big[S_i, S_k\big] C + A\big[B, S_k\big] S_i C, \tag{1.4}$$

where each of the matrices A, B and C can be a product of any number of spin matrices. The term involving the commutator of S_i and S_k contains, through Eq. (1.1), one less spin matrix compared to the terms on the left side of this equation. The other terms on the right-hand side, i.e. those involving commutators with B, can be reduced by application of the identity

$$\big[X, YZ\big] = \big[X, Y\big] Z + Y\big[X, Z\big] \tag{1.5}$$

which is valid for any three objects X, Y and Z for which multiplication is associative. By repeated application of this identity, one attains a stage where Eq. (1.1) can be applied, thereby reducing the number of spin matrices in the expression. We have therefore shown that any product of an arbitrary number of spin matrices can be expressed with fewer number of spin matrices as long as the original product is antisymmetric with respect to the interchange of at least one pair of indices.

It therefore remains to be seen whether the same kind of reduction is possible for other kinds of products of spin matrices, which are not antisymmetric under the interchange of any pair of indices. It suffices to examine the possibility for completely symmetric products, because any product can be written in terms of symmetric and antisymmetric combinations.

We will show that reduction identities exist among completely symmetric combinations of spin matrices. However, unlike Eqs. (1.1) and (1.3) which are obeyed by all representations of the spin matrices, the identities that follow depend on the dimensionality of the representations. In order to present these identities, we will use curly brackets to denote symmetric combination of any number of spin matrices. Thus, for example, with two spin matrices, the symmetric product is denoted by

$$\{S_i S_j\} \equiv S_i S_j + S_j S_i\,, \tag{1.6}$$

which is just the anticommutator. With three matrices, we define

$$\{S_i S_j S_k\} \equiv S_i S_j S_k + S_j S_k S_i + S_k S_i S_j + S_i S_k S_j + S_k S_j S_i + S_j S_i S_k$$
$$= \{S_i S_j\} S_k + \{S_j S_k\} S_i + \{S_k S_i\} S_j\,. \tag{1.7}$$

The generalization is obvious. For example, the symmetric combination with four spin matrices is given by

$$\{S_i S_j S_k S_l\} \equiv \{S_i S_j S_k\} S_l + \{S_j S_k S_l\} S_i + \{S_k S_l S_i\} S_j + \{S_l S_i S_j\} S_k. \tag{1.8}$$