AN EQUIVALENCE OF FINSLERIAN RELATIVISTIC THEORIES

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In Lorentz–Finsler geometry it is natural to define the Finsler Lagrangian over a cone (Asanov's approach) or over the whole slit tangent bundle (Beem's approach). In the former case one might want to add differentiability conditions at the boundary of the (timelike) cone in order to retain the usual definition of lightlike geodesics. It is shown here that if this is done then the two theories coincide, namely the 'conic' Finsler Lagrangian is the restriction of a slit tangent bundle Lagrangian. Since causality theory depends on curves defined through the future cone, this work establishes the essential uniqueness of (sufficiently regular) Finsler spacetime theories and Finsler causality.

Keywords: Finsler gravity, Finslerian relativity, indicatrices, affine metric.

1. Introduction

In this work we shall establish that any sufficiently regular Lorentz–Finsler Lagrangian defined over a cone $\mathscr{L} \le 0$ can be extended to the whole slit tangent bundle (Theorem 1). This result is expected to be useful in both Lorentz–Finsler geometry and in Finslerian theories of gravity since it implies the equivalence of apparently different approaches. I am referring to the works which followed Asanov's conic approach [1–6] and to the works which followed Beem's slit tangent bundle approach [7–15], and which can be considered, thanks to this work, as studies of the same theory. Some Finslerian approaches which cannot be easily comprised in this unified theory include [16–18].

The mathematical methods used in this work belong to Finsler geometry of indicatrices, affine differential geometry and analysis of convex functions [19–21]. The reader is referred to [13] for a more physically oriented bibliography on Finsler spacetime theories including proposals that have been advanced for the field equations.

Let M be a paracompact, Hausdorff, connected, $(n+1)$-dimensional manifold. Let $\{x^\mu\}$ denote a local chart on M and let $\{x^\mu, v^\nu\}$ be the induced local chart on TM. We start giving a quite general setting for Finsler spacetime theory, which we call the *rough model*.

Let Ω be a subbundle of the slit tangent bundle, $\Omega \subset TM\backslash 0$, such that Ω_x is an open sharp convex cone for every x. A *Finsler Lagrangian* is a map $\mathscr{L}\colon \Omega \to \mathbb{R}$

which is positive homogeneous of degree two in the fiber coordinates,

$$\mathscr{L}(x, sv) = s^2 \mathscr{L}(x, v), \qquad \forall s > 0.$$

It is assumed that the fiber dependence is $C^2(\Omega)$, that $\mathscr{L} < 0$ on Ω and that $\mathscr{L}$ can be continuously extended setting $\mathscr{L} = 0$ on $\partial\Omega$. The metric is defined as the Hessian of $\mathscr{L}$ with respect to the fibers,

$$g_{\mu\nu}(x, v) = \frac{\partial^2 \mathscr{L}}{\partial v^\mu \partial v^\nu},$$

and in index-free notation will be denoted with g_v to stress the dependence on the fiber variables. This Finsler metric provides a map $g\colon \Omega \to T^*M \otimes T^*M$. The manifold $(M, \mathscr{L})$ is called a *Finsler spacetime* whenever g_v is Lorentzian, namely of signature $(-, +, \cdots, +)$. By positive homogeneity we have $\mathscr{L} = \frac{1}{2} g_v(v, v)$ and $\mathrm{d}\mathscr{L} = g_v(v, \cdot)$. The usual Lorentzian–Riemannian case is obtained for $\mathscr{L}$ quadratic in the velocities.

A vector $v \in \bar{\Omega}$ is said to be future timelike, lightlike, or spacelike depending on the sign of $\mathscr{L}(x, v)$, respectively negative, zero, or positive. We denote the sets of these vectors with $I^+ = \Omega$, $E^+ = \partial\Omega$ and $J^+ = \bar{\Omega}$, respectively. The observer space (indicatrix), or velocity space, is $\mathscr{I}_x^- = \{v \in T_x M \colon 2\mathscr{L}(x, v) = -1\}$. The condition $\mathscr{L}(x, v) \to 0$ for $v \to \partial\Omega$ assures that the observer space is fully contained in I_x^+. In particular, it is noncompact and asymptotic to E_x^+. Observe that the Finsler Lagrangian is defined just over a subset of the slit tangent bundle as pioneered by Asanov [2].

Beem's definition of Finsler spacetime is more demanding [7], as in his approach $\mathscr{L}$ is defined over $TM \backslash 0$. In this case the Finsler Lagrangian is *reversible* if $\mathscr{L}(x, -v) = \mathscr{L}(x, v)$. Now a selection has to be made of *future* timelike cone (for reversible Lagrangians there is always a time oriented double covering). It is known [7, 11, 13, 22] that the set of timelike vectors consists of the union of disjoint open sharp convex cones. In [13] we proved that for reversible Lagrangians of Beem's type and for $n \geq 2$, there are indeed two timelike cones at each point, exactly as in Lorentzian geometry.[1] The Finsler spacetime in Beem's sense is then a time orientable Lorentz–Finsler manifold.

Of course, Beem's spacetime can be regarded as a particular case of the rough setting where Ω can be identified with the future cone I^+. The attractive features of Beem's approach stand on the C^2 differentiability of the Lagrangian at the boundary of $\Omega = I^+$. This makes it possible to define lightlike geodesics. Also the standard theory of Finsler connections [14, 23–27], being based on the slit tangent bundle, is well adapted to Beem's framework.

One could try to improve the rough theory, while refraining from adopting Beem's approach, through the assumption of Beem's differentiability conditions at the boundary of the lightlike cone as done in [6]. However, this strategy does not lead to a different physical theory since we prove the following result.

[1] John Beem investigated this problem and believed to have found a counterexample [22] which, under closer inspection, can be shown to be incorrect.