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PARAMETRIC QUANTUM SEARCH ALGORITHM AS QUANTUM WALK: A QUANTUM SIMULATION

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Parametric quantum search algorithm (PQSA) is a form of quantum search that results by relaxing the unitarity of the original algorithm. PQSA can naturally be cast in the form of quantum walk, by means of the formalism of oracle algebra. This is due to the fact that the completely positive trace preserving search map used by PQSA, admits a unitarization (unitary dilation) a la quantum walk, at the expense of introducing auxiliary quantum coin-qubit space. The ensuing QW describes a process of spiral motion, chosen to be driven by two unitary Kraus generators, generating planar rotations of Bloch vector around an axis. The quadratic acceleration of quantum search translates into an equivalent quadratic saving of the number of coin qubits in the QW analogue. The associated to QW model Hamiltonian operator is obtained and is shown to represent a multi-particle long-range interacting quantum system that simulates parametric search. Finally, the relation of PQSA-QW simulator to the QW search algorithm is elucidated.

Keywords: quantum search, quantum walk, quantum simulation, CP map, Lie algebra.

1. Introduction

General setting. Let us recall the fast quantum search algorithm [1–4], with searching matrix $U_G = -U J_s U^\dagger J_x$, used to search for $1 \leq k \ll N$ entries encoded in vectors $\{|j\rangle \mid 1 \leq j \leq k\}$ among N orthonormal others, that span the complex Hilbert space of an unsorted quantum database of vectors $l_2(\Delta) = \text{span}\{|i\rangle\}_{i=1}^N$, all enumerated by index set $\Delta = \{1, 2, \ldots, N\}$. Here U is an undetermined general $U(N)$ unitary matrix, and $J_s = \mathbf{1} - 2|s\rangle\langle s|$, and $J_x = \mathbf{1} - 2|x\rangle\langle x|$ are reflection operators with respect to vectors $|s\rangle$ and $|x\rangle$. Vector

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle$$

is the uniform superposition state of all database vectors,

$$|x\rangle = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} |j\rangle$$

the uniform superposition of all marked vectors, and

$$|x^{\perp}\rangle = \frac{1}{\sqrt{N-k}} \sum_{j=1}^{k} |j\rangle$$

is the uniform superposition of all unmarked vectors in $l_2(\Delta) = \text{span}\{|i\rangle\}_{i=1}^{N}$, also $\mathcal{V}_x = \text{span}\{|x\rangle, |x^{\perp}\rangle\} \approx \mathbb{C}^2$, where $U \in U(N)$ (see Appendix A for definitions). Remarkably, while classical search requires $O(N/k)$ trials for finding the target item $|x\rangle$, the quantum algorithm is quadratically faster since it determines the items after only $O(\sqrt{N/k})$ queries (see [5] for a recent review and e.g. [6, 7], for some recent developments).

Several authors have investigated the influence on algorithm's complexity of random imperfections both in diffusion operation generated by reflection J_s, and in black-box query generated by reflection J_x, in models that preserve the unitary character of search, i.e. the pureness of density operator [8–10]. Also more recent works [12, 13], have considered search which takes into account possible errors in reporting the correct marked entry with certain failure probabilities. The errors can concern all or some of the marked items, and are modelled by randomization of the oracle operator J_x, resulting into mixed density matrices. In all these generalized search models the account of noise is destructive for the effectiveness of search (for a summary see e.g. [11]). Designated under the common name *parametric quantum search* (PQS), various possibilities of introducing errors, inaccuracies and noise in the algorithm are summarized in the diagrammatic display below.

	random diffusion J_s	random oracle J_x
pure state	[8–10]	[8–10]
mixed state	[18, 19]	[12, 13]

Examples of parametric search models.

The row of the diagram labels the states of search system (pure, mixed), resulting after the introduction of errors, inaccuracies or noise, in either of the two constituents of the algorithm (diffusion process, oracle query), labelling the columns. The randomization of the diffusion operator J_s as well as the randomization of the oracle operator J_x, is introduced in various models with various theoretical and implementational motivations and justifications. Actually these randomizations introduce a number of extra parameters into the algorithm which in some generalized models becomes an open quantum system [14–16], whose state is a density matrix evolving in discrete time. Also we notice that all these generalized search models preserve a basic feature of initial algorithm, namely the $2D$ sub-space $\mathcal{V}_x = \text{span}\{|x\rangle, |x^{\perp}\rangle\} \approx \mathbb{C}^2$; any failure of the qubit registers encoding states $|x\rangle$, $|x^{\perp}\rangle$, e.g. by loosing some qubits, would be an additional source of errors that would go beyond the PQSA scheme discussed here (for a recent study of such cases see [17], and reference therein).