ASYMPTOTIC FORMULA FOR QUANTUM HARMONIC OSCILLATOR TUNNELING PROBABILITIES

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Using simple methods of asymptotic analysis it is shown that for a quantum harmonic oscillator in n-th energy eigenstate the probability of tunneling into the classically forbidden region obeys an unexpected but simple asymptotic formula: the leading term is inversely proportional to the cube root of n.

Keywords: quantum harmonic oscillator, turning points, tunneling probabilities, asymptotic, Hermite polynomials, Airy function.

1. Introduction

Consider the quantum harmonic oscillator in dimensionless variables, where the Hamiltonian, in terms of position and momentum operators $\hat{x}, \hat{p}$, satisfying $[\hat{x}, \hat{p}] = i$, is given by the formula

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2).$$

Its normalized eigenstates, when represented as square integrable functions of the position variable x are

$$\psi_n(x) = \pi^{-1/4}(2^n n!)^{-1/2} H_n(x)\, e^{-x^2/2},$$

H_n are Hermite's polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

The corresponding probability densities $P_n(x)$ are then given by $|\psi_n(x)|^2$, i.e.

$$P_n(x) = a_n (H_n(x)\, e^{-x^2/2})^2,$$

where

$$a_n = \frac{1}{\pi^{1/2} 2^n n!}, \tag{1}$$

so that the normalization condition holds,

$$\int_{-\infty}^{\infty} P_n(x)dx = 1.$$

The functions $P_n(x)$ are symmetric with respect to the origin $x = 0$. The classical turning points are at $x = \pm\sqrt{2n+1}$, thus the probabilities of quantum tunneling into the classically forbidden region are given by the formula

$$P_{n,\text{tun}} = 2\int_{\sqrt{2n+1}}^{\infty} P_n(x)\, dx = 2a_n Q_n, \tag{2}$$

where

$$Q_n = \int_{\sqrt{2n+1}}^{\infty} \left(H_n(x)\, e^{-x^2/2}\right)^2\, dx. \tag{3}$$

The probability distributions $P_n(x)$ are usually represented (see, for example, [1, p. 168]) as in Fig. 1

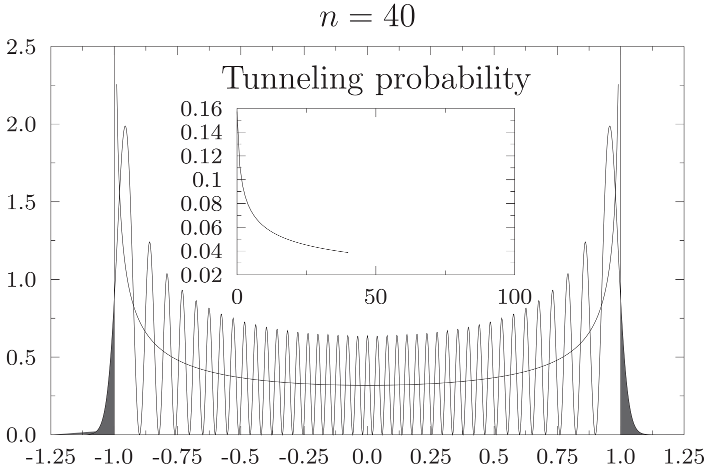

Fig. 1. Probability distribution $P_{40}(x)$, with the classical turning points scaled to $x = \pm 1$, and the curve $P_{n,\text{tun}}$ for n from 0 to 40. The black U-shaped curve represents classical probability distribution $P_{\text{class}}(x) = 1/(\pi\sqrt{1-x^2})$ for an oscillator with known energy and unknown phase. Shadowed tails beyond the classical turning points are responsible for nonzero quantum tunneling probabilities.

The tunneling probabilities $P_{n,\text{tun}}$ are rarely discussed in textbooks on quantum mechanics. Sometimes they are discussed in quantum chemistry textbooks. For instance in [7, p. 92] the first three are calculated (though the second one with an error: it should be 0.1116 instead of 0.116). In [8, p. 66-67; p. 98-99 in 2nd ed. (2014)] $P_{0,\text{tun}}$ is calculated with a comment: "*The probability of being found in classically forbidden regions decreases quickly with increasing* n, *and vanishes entirely as n approaches infinity*. " Yet the question of "how quickly" is left open