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ON THE CHIRALITY OF A DISCRETE DIRAC–KÄHLER EQUATION

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We discuss a discrete analogue of the Dirac–Kähler equation in which chiral properties of the continuum counterpart are captured. We pay special attention to a discrete Hodge star operator. To build such an operator combinatorial construction of a double complex is used. We describe discrete exterior calculus operations on a double complex and obtain the discrete Dirac–Kähler equation using these tools. Self-dual and anti-self-dual discrete inhomogeneous forms are presented. The chiral invariance of the massless discrete Dirac–Kähler equation is shown. Moreover, in the massive case we prove that a discrete Dirac–Kähler operator flips the chirality.

Keywords: Dirac–Kähler equation, chirality, Hodge star operator, difference equations, discrete models.

1. Introduction

This work is a direct continuation of the paper [17] in which we have constructed a new discrete analogue of the Dirac–Kähler equation. In [17] we have proposed a geometric discretisation scheme based on the formalism of differential forms and we have shown that many of the algebraic relations amongst the Hodge star operator $*$, the exterior product $\wedge$, the differential d, and the adjoint δ of d that hold in the smooth setting also hold in the discrete case. This approach was originated by Dezin [6]. There are alternative geometric discretisation schemes based on the use of the differential forms language [1, 3, 4, 7, 8, 14, 16, 19]. Difficulties of the discretisation of the Hodge operators have been described by several authors [3, 14, 16, 21, 22, 24].

In this paper we are going to discuss a decomposition of the discrete Dirac–Kähler equation into its self-dual and anti-self-dual parts. The reason that we did not consider this problem in [17] is that the Hodge star operator $*$ and its discrete analogue differ slightly. In the continuum case the operator $(*)^2$ is either an involution or antiinvolution while in the discrete model $(*)^2$ is equivalent to a shift with corresponding sign. This is one of the main distinctive features of the formalism [17] as compared to the continuum case. Now we define a discrete star operator using a combinatorial double complex construction. In this way we obtain the operator which is more like its continuum analogue since $(*)^2 = \pm I$, where I is the identity operator. At the same time in the double complex we will use discrete analogues of differential forms, d, and $\wedge$ defined in [17]. The discrete star operator proposed here still preserves the Lorentz metric structure in our discrete model. It makes possible to define a discrete analogue of δ by using an inner product of discrete forms (cochains) which imitate the continuum case.

From the physics point of view the self-dual and anti-self-dual fields of the Dirac theory correlate with chiral fermions. It is well known that the chirality is an important feature of the Dirac theory. However, in the lattice formulations of fermions the chirality problem is one of the most notorious. This problem deals with breaking chiral symmetry on the lattice [5, 9, 12, 13, 20], see also more recent work [10]. For the Dirac–Kähler equation on the lattice this difficulty was discussed first by Rabin [14]. In [2], a discretisation scheme in the Dirac–Kähler setting was proposed in which the chirality is captured on the lattice. In [23], Watterson describes an exact chiral symmetry for Dirac–Kähler fermions using a simplicial complex and its dual of the geometric discretization. This approach is based on the Whitney map from cochains to differential forms and the de Rham map, which maps the other way. Watterson defines two discrete star operators using the duality map between two complexes.

In this paper we derive a purely abstract discrete model of the Dirac–Kähler equation. The discrete analogues of the basic operations are defined without using the Whitney and the de Rham maps and this is the significant difference of the proposed discretisation scheme. However, the algebraic relations between $\wedge$, d, $*$, and δ are preserved by their discrete counterparts (for example, a difference analog of d satisfies the Leibniz rule). Formally, the exterior calculus structure is captured in the discrete case. Moreover, the discrete Dirac–Kähler equation is expressed clearly in terms of difference equations and this is an advantage of our discrete model, which can be used for numerical calculations. A contradiction with the Nielsen–Ninomiya theorem [25–27] is avoided, since we use the Dirac–Kähler formulation and we have a close analogy between the discrete model and differential geometry in our method. This fact has been mentioned in [2].

Chiral fermions can be described by using the fifth gamma matrix: $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. The projection operators $P_L = \frac{I-\gamma^5}{2}$ and $P_R = \frac{I+\gamma^5}{2}$ decompose any Dirac field into its left-handed and right-handed parts. Rabin [14] pointed out that in the language of differential forms the chiral symmetry is a rotation of mixing forms with their duals and the Hodge star operator plays a central role here. However, the operator $*$ is somewhat different from γ^5. Therefore, we introduce a modified star operator which plays the role of γ^5 in our discrete model. The self-dual and anti-self-dual discrete inhomogeneous forms with respect to this new star operator are considered. From the viewpoint presented here this allows us to deal with chirality. We show that, just as in the continuum case, a discrete massless Dirac–Kähler equation admits the chiral invariance. We prove also that in the massive case a discrete analog of the Dirac–Kähler operator flips the chirality.

We first briefly review some definitions and basic notation on the Dirac–Kähler equation [11, 14]. Let $M = \mathbb{R}^{1,3}$ be a Minkowski space with the metric signature

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