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Polar rotation angle identifies elliptic islands in unsteady dynamical systems

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HIGHLIGHTS

- Rigid body rotation of phase space elements extracted from flow gradient.
- Polar rotation angle (PRA) is introduced to identify invariant tori.
- Applications to vortex detection are illustrated.

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ABSTRACT

We propose rotation inferred from the polar decomposition of the flow gradient as a diagnostic for elliptic (or vortex-type) invariant regions in non-autonomous dynamical systems. We consider here two- and three-dimensional systems, in which polar rotation can be characterized by a single angle. For this polar rotation angle (PRA), we derive explicit formulas using the singular values and vectors of the flow gradient. We find that closed level sets of the PRA reveal elliptic islands in great detail, and singular level sets of the PRA uncover centers of such islands. Both features turn out to be objective (frame-invariant) for two-dimensional systems. We illustrate the diagnostic power of PRA for elliptic structures on several examples.

1. Introduction

Complex dynamical systems exhibit a mixture of chaotic and coherent behavior in their phase space. The latter manifests itself in coherent islands of regular behavior surrounded by a chaotic background flow. The best known classic examples of such islands are formed by Kolmogorov–Arnold–Moser (KAM) tori, composed of quasi-periodic trajectories in Hamiltonian systems (see, e.g., [1,2]). Outside elliptic regions filled by such tori, chaotic trajectories dominate the dynamics. For steady, time-periodic and quasi-periodic flows, the techniques of KAM theory help in visualizing elliptic regions (see, e.g., [3,4] for recent examples).

Even more intriguing is the existence of similar elliptic islands in turbulent fluid flow, as broadly confirmed by experiments and numerical simulations (see, e.g., [5,6]). Just as KAM islands, coherent vortices capture trajectories and keep them out of chaotic mixing zones. Unlike KAM tori, however, coherent vortices are composed of trajectories that are generally not recurrent in any frame. During their finite time of existence, these coherent vortices traverse without filamentation but also without displaying any particular periodic or quasiperiodic pattern. Still, we generally refer to such regions here as elliptic, as they mimic the dynamic role of elliptic islands occupied by classic KAM tori.

Eulerian approaches to describing elliptic islands seek domains where rotation dominates the instantaneous velocity field. At the simplest level, this involves locating regions of closed streamlines, high enough vorticity or low enough pressure (cf. [7,8] for reviews). Such domains reveal instantaneous velocity field features at a low cost, but are unable to frame long-term material coherence exhibited by trajectories. In addition, the results from these instantaneous approaches depend on the choice of scalar thresholds and on the frame of reference.

More sophisticated Eulerian principles for elliptic regions seek sets of points where rotation dominates strain (see, e.g., [9–12,7,13], and also Jeong and Hussain [7] and Haller [8] for

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reviews). These principles infer both rotation and strain from the instantaneous velocity gradient, thereby rendering the results Galilean invariant. The elliptic regions they provide, however, still change under rotations of the frame. Since truly unsteady flows have no distinguished frame of reference [14], frame-dependence in the detection of vortical structures is an impediment. Indeed, the available measurement velocity data of geophysical flows is often given in a rotating frame to begin with, and no optimal frame is known a priori for structure detection. More importantly, no mathematical relationship is known (or likely to exist) between instantaneous rotation–strain principles and material coherence over extended time intervals.

In contrast, Lagrangian approaches to elliptic islands seek to identify regions where trajectories stay close for longer periods. These approaches can roughly be divided into three categories: geometric, set-based and diagnostic methods. The geometric methods identify elliptic domain boundaries as spacial closed material lines showing no filamentation [15–17] or curvature change [18]. Set-based methods partition the phase space into almost invariant subsets (see [19,20] and references therein). While the boundaries of such sets may undergo filamentation, the overall subsets remain largely coherent. Finally, diagnostic approaches propose Lagrangian scalar fields whose features are expected to distinguish mixing regions from coherent ones [21–26]. These Lagrangian methods do not return identical results and are not backed by specific mathematical results on the features they highlight. In fact, the material invariance of the extracted vortical boundaries is only guaranteed in the case of the geodesic approach of Haller and Beron-Vera [15] and Haller [17].

The Lagrangian methods listed above focus on stretching or lack thereof. In contrast, very few Lagrangian diagnostics target rotation, even though sustained and coherent rotation is perhaps the most striking feature of trajectories forming elliptic islands. One of the few exceptions targeting material rotation is the finite-time rotation number (FTRN), developed to detect hyperbolic (i.e., repelling or attracting as opposed to vortical) structures through its ridges [27]. The FTRN assumes that the dynamical system is defined via an iterated map with an annular phase space. For dynamical systems with general time dependence and non-annular phase space, however, this approach is not applicable. This also means that the approach is frame-dependent, given that translations and rotations will generally destroy the time-periodicity of a dynamical system.

Another Lagrangian diagnostic involving a consideration of rotation is the mesocronic analysis of Mezić et al. [25]. This approach offers a formal extension of the Okubo–Weiss principle from the velocity gradient to the flow gradient, classifying an initial condition as elliptic if the flow gradient has complex eigenvalues at that point. The mesoelliptic diagnostic is efficient to compute and has been shown to mark vortical regions in several cases. The complex eigenvalues of a finite-time flow map, however, have no known mathematical relationship with elliptic islands in flows with general time dependence. Accordingly, some annular subsets of classic elliptic domains fail the test of meso-ellipticity even in steady flows (cf. [25], Fig. 1).

Here we propose a mathematically precise assessment of material rotation, the polar rotation angle (PRA), as a new diagnostic for elliptic islands in two- and three-dimensional flows. The PRA is the angle of the rigid-body rotation component obtained from the classic polar decomposition of the flow gradient into a rotational and a stretching factor. We show how the PRA can readily be computed from invariants of the flow gradient and the Cauchy–Green strain tensor. Level sets of the PRA turn out to be objective (frame-invariant) in planar flows. We find that these level sets reveal the internal structure of elliptic islands in great detail at a relatively low computational cost. We also find that local extrema of the PRA mark elliptic island centers suitable for automated vortex tracking in Lagrangian fluid dynamics.

2. Preliminaries

2.1. Set-up

Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^3,\ t \in I \subset \mathbb{R}, \tag{1}$$

with the corresponding flow map

$$\mathbf{F}_{t_0}^t : \mathcal{D} \to \mathcal{D} \\ \mathbf{x}_0 \mapsto \mathbf{x}(t; t_0, \mathbf{x}_0), \tag{2}$$

the diffeomorphism that takes the initial condition $\mathbf{x}_0$ to its time-t position $\mathbf{x}(t; t_0, \mathbf{x}_0)$ under system (1). Here, $\mathcal{D}$ denotes the phase space and I is a finite time interval of interest.

The deformation gradient $\nabla\mathbf{F}_{t_0}^t$ governs the infinitesimal deformations of the phase space $\mathcal{D}$. In particular, an initial perturbation $\boldsymbol{\xi}$ at point $\mathbf{x}_0$ and time t_0 is mapped, under the system (1), to $\nabla\mathbf{F}_{t_0}^t(\mathbf{x}_0)\boldsymbol{\xi}$ at time t. We also define the *right Cauchy–Green strain tensor*,

$$\mathbf{C}_{t_0}^t := \left[\nabla\mathbf{F}_{t_0}^t\right]^\top \nabla\mathbf{F}_{t_0}^t : \mathbf{x}_0 \mapsto \mathbf{C}_{t_0}^t(\mathbf{x}_0), \tag{3}$$

where the symbol $\top$ denotes matrix transposition. The tensor $\mathbf{C}_{t_0}^t(\mathbf{x}_0)$ is symmetric and positive definite. Therefore, it has an orthonormal set of eigenvectors $\{\boldsymbol{\xi}_1(\mathbf{x}_0), \boldsymbol{\xi}_2(\mathbf{x}_0), \boldsymbol{\xi}_3(\mathbf{x}_0)\}$. The corresponding eigenvalues $0 < \lambda_1(\mathbf{x}_0) \le \lambda_2(\mathbf{x}_0) \le \lambda_3(\mathbf{x}_0)$ therefore satisfy

$$\mathbf{C}_{t_0}^t(\mathbf{x}_0)\boldsymbol{\xi}_i(\mathbf{x}_0) = \lambda_i(\mathbf{x}_0)\boldsymbol{\xi}_i(\mathbf{x}_0), \quad i \in \{1, 2, 3\}, \tag{4}$$

$$\langle \boldsymbol{\xi}_j(\mathbf{x}_0), \boldsymbol{\xi}_k(\mathbf{x}_0)\rangle = 0, \quad j, k \in \{1, 2, 3\},\ j \neq k, \tag{5}$$

with $\langle\cdot,\cdot\rangle$ denoting the Euclidean inner product. For notational simplicity, we omit the dependence of the eigenvalues and eigenvectors on t_0 and t.

2.2. Polar decomposition

Any square matrix admits a factorization into the product of a unitary matrix with a symmetric positive-semidefinite matrix [28]. When the square matrix is nonsingular, such as $\nabla\mathbf{F}_{t_0}^t$, then the symmetric factor in the decomposition is positive definite.

Specifically, the deformation gradient $\nabla\mathbf{F}_{t_0}^t$ admits a unique decomposition of the form

$$\nabla\mathbf{F}_{t_0}^t = \mathbf{R}_{t_0}^t\mathbf{U}_{t_0}^t, \tag{6}$$

where the 3×3 matrices $\mathbf{R}_{t_0}^t$ and $\mathbf{U}_{t_0}^t$ have the following properties [28–30]:

1. The rotation tensor $\mathbf{R}_{t_0}^t$ is proper orthogonal, i.e.,
$$\left(\mathbf{R}_{t_0}^t\right)^\top\mathbf{R}_{t_0}^t = \mathbf{R}_{t_0}^t\left(\mathbf{R}_{t_0}^t\right)^\top = \mathbf{I}, \qquad \det\mathbf{R}_{t_0}^t = 1.$$
2. The right stretch tensor $\mathbf{U}_{t_0}^t$ is symmetric and positive-definite, satisfying
$$\left[\mathbf{U}_{t_0}^t\right]^2 = \mathbf{C}_{t_0}^t. \tag{7}$$
3. The eigenvalues of $\mathbf{U}_{t_0}^t$ are $\sqrt{\lambda_k}$ with corresponding eigenvectors $\boldsymbol{\xi}_k$:
$$\mathbf{U}_{t_0}^t(\mathbf{x}_0)\boldsymbol{\xi}_k(\mathbf{x}_0) = \sqrt{\lambda_k(\mathbf{x}_0)}\boldsymbol{\xi}_k(\mathbf{x}_0), \quad k = 1, 2, 3. \tag{8}$$
4. The time derivative of the rotation tensor satisfies
$$\dot{\mathbf{R}}_{t_0}^t = \left(\mathbf{W}(\mathbf{x}(t), t) - \frac{1}{2}\mathbf{R}_{t_0}^t\left[\dot{\mathbf{U}}_{t_0}^t\left(\mathbf{U}_{t_0}^t\right)^{-1} - \left(\mathbf{U}_{t_0}^t\right)^{-1}\dot{\mathbf{U}}_{t_0}^t\right]\left(\mathbf{R}_{t_0}^t\right)^\top\right)\mathbf{R}_{t_0}^t, \tag{9}$$