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Positive and necklace solitary waves on bounded domains

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HIGHLIGHTS

- We present new solitary wave solutions of the two-dimensional NLS on bounded domains.
- These necklace solitary waves become unstable well below the critical power for collapse.
- On the annulus they have a second stability regime well above the critical power.
- We introduce a non-spectral variant of Petviashvili's renormalization method.

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ABSTRACT

We present new solitary wave solutions of the two-dimensional nonlinear Schrödinger equation on bounded domains (such as rectangles, circles, and annuli). These multi-peak "necklace" solitary waves consist of several identical positive profiles ("pearls"), such that adjacent "pearls" have opposite signs. They are stable at low powers, but become unstable at powers well below the critical power for collapse P_{cr}. This is in contrast with the ground-state ("single-pearl") solitary waves on bounded domains, which are stable at any power below P_{cr}.

On annular domains, the ground state solitary waves are radial at low powers, but undergo a symmetry breaking at a threshold power well below P_{cr}. As in the case of convex bounded domains, necklace solitary waves on the annulus are stable at low powers and become unstable at powers well below P_{cr}. Unlike on convex bounded domains, however, necklace solitary waves on the annulus have a second stability regime at powers well above P_{cr}. For example, when the ratio of the inner to outer radii is 1:2, four-pearl necklaces are stable when their power is between $3.1P_{cr}$ and $3.7P_{cr}$. This finding opens the possibility to propagate localized laser beams with substantially more power than was possible until now.

The instability of necklace solitary waves is excited by perturbations that break the antisymmetry between adjacent pearls, and is manifested by power transfer between pearls. In particular, necklace instability is unrelated to collapse. In order to compute numerically the profile of necklace solitary waves on bounded domains, we introduce a non-spectral variant of Petviashvili's renormalization method.

1. Introduction

The nonlinear Schrödinger equation (NLS) in free space

$$i\psi_z(z,x,y) + \Delta\psi + |\psi|^2\psi = 0, \quad -\infty < x, y < \infty, \ z > 0, \quad (1a)$$

$$\psi(0,x,y) = \psi_0(x,y), \quad -\infty < x, y < \infty \quad (1b)$$

is one of the canonical nonlinear equations in physics. In nonlinear optics it models the propagation of intense laser beams in a bulk Kerr medium. In this case, z is the axial coordinate in the direction of propagation, x and y are the spatial coordinates in the transverse plane, $\Delta\psi := \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi$ is the diffraction term, and $|\psi|^2\psi$ describes the nonlinear Kerr response of the medium. For more information on the NLS in nonlinear optics and on NLS theory in free space and on bounded domains, see the recent book [1].

In some applications, it is desirable to propagate laser beams over long distances. In theory, this can be done by the NLS solitary waves $\psi_{sw} = e^{i\mu z}R_\mu(x,y)$, where R_μ is a solution of

$$\Delta R(x,y) - \mu R + |R|^2R = 0, \quad -\infty < x, y < \infty. \quad (2)$$

Unfortunately, the solitary wave solutions of (1) are unstable, so that when perturbed, they either scatter (diffract) as $z \to \infty$, or collapse at a finite distance $Z_c < \infty$.

In order to mitigate this "dual instability" limitation, Soljacic, Sears, and Segev [2] proposed in 1998 to use a necklace configuration that consists of n identical beams ("pearls") that are located

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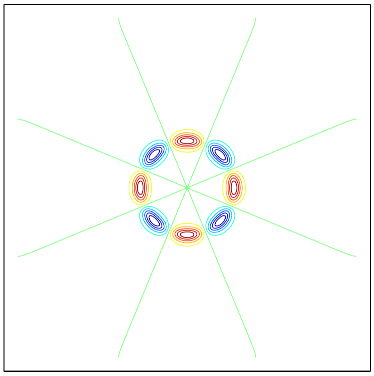

Fig. 1. A circular necklace beam with 8 pearls (beams). Adjacent pearls are identical but have opposite phases. The electric field vanishes on the rays (solid lines) between adjacent pearls.

along a circle at equal distances, such that adjacent beams are out of phase (i.e., have opposite signs), see Fig. 1. The idea behind this setup is that the repulsion between adjacent out-of-phase beams resists the diffraction of each beam, and thus slows down its expansion. Necklace beams in a Kerr medium were first observed experimentally by Grow et al. [3]. Necklace beams were also studied in [4–7]. Recently, Jhajj et al. used a necklace-beam configuration to set up a thermal waveguide in air [8].

As we shall see, there are no necklace solitary wave solutions of the free-space NLS (1). Thus, in a bulk medium all necklace beams ultimately collapse or scatter. Yang et al. [7] showed theoretically and experimentally that solitary necklace solutions can exist in a bulk medium with an optically-induced photonic lattice. Because of the need to induce a photonic lattice, however, this approach is not applicable to propagation is a Kerr medium. In this study we show that necklace solitary waves exist in a Kerr medium, provided the beam is confined to a bounded domain. This setup corresponds to propagation in hollow-core fibers, and is therefore relatively easy to implement experimentally.

In hollow-core fibers, beam propagation can be modeled by the NLS on a bounded domain

$$i\psi_z(z,x,y) + \Delta\psi + |\psi|^2\psi = 0, \quad (x,y) \in D,\ z > 0, \tag{3a}$$

subject to an initial condition

$$\psi_0(0,x,y) = \psi_0(x,y), \quad (x,y) \in D, \tag{3b}$$

and a Dirichlet boundary condition at the fiber wall

$$\psi(z,x,y) = 0, \quad (x,y) \in \partial D,\ z \geq 0. \tag{3c}$$

Here $D \subset \mathbb{R}^2$ is the cross section of the fiber, which is typically a circle of radius ρ, denoted henceforth by B_ρ.

Eq. (3) admits the solitary waves $\psi_{sw} = e^{i\mu z}R_\mu(x,y)$, where R_μ is a solution of

$$\Delta R(x,y) - \mu R + |R|^2 R = 0, \quad (x,y) \in D, \tag{4a}$$

$$R(x,y) = 0, \quad (x,y) \in \partial D. \tag{4b}$$

In free space, the solitary-wave profile [i.e., the solution of (2)] represents a perfect balance between the focusing nonlinearity and diffraction. On a bounded domain, the reflecting boundary "works with" the focusing nonlinearity and "against" the diffraction. In fact, the reflecting boundary can support finite-power solitary waves even in the absence of a focusing linearity. These linear modes are solutions of the eigenvalue problem

$$\Delta Q(x,y) = \mu Q, \quad (x,y) \in D, \qquad Q(x,y) = 0, \quad (x,y) \in \partial D. \tag{5}$$

Of most importance is the first eigenvalue of (5) and its corresponding positive eigenfunction, which we shall denote by μ_{lin} and $Q^{(1)}$, respectively.

Solitary wave solutions of (3) were studied by Fibich and Merle [9], Fukuizumi, Hadj Selem, and Kikuchi [10], and Noris, Tavares, and Verzini [11], primarily when D is the unit circle B_1 and R_μ is radial, i.e., $R_\mu = R_\mu(r)$, $r = \sqrt{x^2+y^2}$, and $0 \leq r \leq 1$. In that case, for any $\mu_{\text{lin}} < \mu < \infty$, there exists a unique positive solution $R_\mu^{(1)}(r)$. This solution is monotonically decreasing in r, and its power $P(R_\mu^{(1)}) := \int_{B_1} |R_\mu^{(1)}|^2\, dxdy$ is monotonically increasing in μ from $P = 0$ at $\mu = \mu_{\text{lin}}+$ to $P = P_{\text{cr}}$ as $\mu \to \infty$. In addition to this ground state, there exist a countable number of excited radial states $\{R_\mu^{(n)}(r)\}_{n=2}^\infty$, which are non-monotone and change their sign inside B_1. These excited states have a unique global maximum at $r = 0$, and additional lower peaks on concentric circles inside B_1.

The excited states $\{R_\mu^{(n)}(r)\}_{n=2}^\infty$ are the two-dimensional radial analog of the excited states of the one-dimensional NLS on an interval, see Eq. (26) below, which were studied by Fukuizumi et al. [10]. In our study here we consider a different type of solitary waves of (3), which attain their global maximum at n distinct points inside D. These necklace solitary waves are thus the non-radial two-dimensional analog of the one-dimensional excited states.

The paper is organized as follows. In Section 2 we briefly consider necklace solutions in $\mathbb{R}^2$, which correspond to propagation in a bulk medium. We illustrate numerically that their expansion is slower than that of single-beam solutions, and provide an informal proof that there are no necklace solitary waves in free space. In Section 3 we briefly review the theory for the NLS on bounded domains. In Section 4 we construct necklace solitary waves with n pearls (peaks), denoted by $R_\mu^{(n)}$, on rectangular, circular, and annular domains. To do that, we first compute the single-pearl (single-peak/ground state) solitary wave of (4), denoted by $R_\mu^{(1)}$, on a square, a sector of a circle, and a sector of an annulus, respectively. Our numerical results for single-pearl solutions of (4) suggest that[1]:

1. $R_\mu^{(1)}$ exists for μ in the range $\mu_{\text{lin}} < \mu < \infty$.
2. As $\mu \to \mu_{\text{lin}}$, $R_\mu^{(1)}$ approaches the positive linear mode $Q^{(1)}$, i.e., $R_\mu^{(1)} \sim c(\mu)Q^{(1)}$, where $c(\mu) \to 0$.
3. As μ increases, $R_\mu^{(1)}$ becomes more localized, the effect of the nonlinearity becomes more pronounced, and that of the reflecting boundary becomes less pronounced.
4. In particular, as $\mu \to \infty$, $R_\mu^{(1)}$ approaches the free-space ground state $R_{\mu,2D}^{(1),\text{free}}$, which is the positive solution of (2).
5. The pearl power $P(R_\mu^{(1)}) := \int |R_\mu^{(1)}|^2\, dxdy$ is monotonically increasing in μ. In particular,
$$\frac{d}{d\mu}P(R_\mu^{(1)}) > 0, \qquad \lim_{\mu\to\mu_{\text{lin}}} P(R_\mu^{(1)}) = 0, \qquad \lim_{\mu\to\infty} P(R_\mu^{(1)}) = P_{\text{cr}}, \tag{6}$$
where
$$P_{\text{cr}} = \int_{\mathbb{R}^2} \left|R_{\mu,2D}^{(1),\text{free}}\right|^2 dxdy$$
is the critical power for collapse.
6. On an annular domain, the ground state solitary waves are radial (ring-type) at low powers, but undergo a *symmetry breaking* into a single-peak profile at a threshold power well below P_{cr}. In particular, Eq. (4) on the annulus,
   (a) does not have a unique positive solution.
   (b) has a positive solution which is not a ground state.

[1] These results are consistent with those obtained for radial positive solitary waves on the circle [9] and for positive one-dimensional solitary waves on an interval [10].