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Mixed dynamics in a parabolic standard map

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HIGHLIGHTS

- A perturbed area-preserving map from the boundary of hyperbolicity has a parabolic orbit.
- Numerics show a channel of measure less than one containing elliptic orbits.
- The complement of the channel has positive measure and positive Lyapunov exponents.

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ABSTRACT

We use numerical and analytical tools to provide arguments in favor of the existence of a family of smooth, symplectic diffeomorphisms of the two-dimensional torus that have both a positive measure set with positive Lyapunov exponent and a positive measure set with zero Lyapunov exponent. The family we study is the unfolding of an almost-hyperbolic diffeomorphism on the boundary of the set of Anosov diffeomorphisms, proposed by Lewowicz.

1. Introduction

Due to an extremely complicated intermixture of regular and chaotic orbits, the problem of the orbit structure of a generic, smooth symplectic map remains mainly open, even for the two-dimensional case. When the map is sufficiently smooth, its phase space typically exhibits both regular dynamics due to invariant KAM curves (for instance, in the neighborhood of elliptic periodic orbits) and seas of chaotic orbits (which numerical investigations indicate can be densely covered by a single orbit). Moreover, such structures are observed – again in numerical simulations – to occur at all scales. All this is well known and shown in many papers, for a review see, e.g., [1]. It is generally agreed that no tools currently exist that allow one to rigorously elucidate the main points of this observed picture [2]. Of course, selected parts of this landscape can be explained; for example, KAM theory provides a proof of the existence of invariant curves near generic elliptic periodic points, and it is known that a generic homoclinic bifurcation gives rise to a transitive set of full Hausdorff dimension [3]. However even for this case, there is essentially no rigorous measure-theoretic characterization of the orbit behavior in the so-called chaotic zones—as depicted in Arnold's famous sketch [4,5].

There has been much study of the destruction of invariant curves, and the resulting transition from regular (quasiperiodic) to irregular (chaotic) behavior, in parameterized families of area-preserving maps. Since a smooth invariant curve is not isolated, its destruction is caused by a loss of smoothness and, at least for twist maps, to the formation of a new, quasiperiodic invariant Cantor set: an Aubry–Mather set [6,7]. In many families, one observes the ultimate destruction of all the invariant circles (of a given homotopy class), and this leads to the study of the "last" invariant curve, and the development of Greene's residue criterion and renormalization theory [8].

At the opposite extreme, the ergodicity and hyperbolicity properties of Anosov diffeomorphisms are well-understood [9]. This extreme of uniform hyperbolicity can be thought of as a complementary limit to integrability: the study of perturbations from "anti-integrability" was initiated in [10]. Aubry's results are based on the consideration of infinitely-degenerate diffeomorphisms and provide proofs of the existence of horseshoes; however, they do not lead to proofs of a positive measure of chaotic orbits.

There have been attempts to understand the dynamics of symplectic diffeomorphisms on the torus beyond the boundary of

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the Anosov maps [11]. Przytycki proved the existence of a curve of diffeomorphisms that cross the Anosov boundary such that, outside the boundary, there is a domain on the torus bounded by a heteroclinic cycle formed by merged separatrices of two saddles that contains a generic elliptic fixed point. The remaining set of positive measure has a nonhyperbolic structure and positive Lyapunov exponent. The drawback of this example is in its infinite codimension in the space of smooth symplectic diffeomorphisms with C^5-topology: the merging of separatrices of saddles is a codimension-infinity phenomenon. Przytycki's family unfolds a smooth, almost-hyperbolic symplectic diffeomorphism of the torus proposed earlier by Lewowicz [12]. This diffeomorphism is a K-system that has positive Lyapunov exponent [13].

This same trick (with the same drawbacks) was used later in [14] to construct symplectic diffeomorphisms arbitrarily close to Lewowicz's almost-hyperbolic map. Smooth symplectic, transitive diffeomorphisms that are K-systems on closed two-dimensional manifolds other than tori were constructed in [15] (see also, [16]). Again it is not clear how these results can be used to understand the orbit structure for a generic diffeomorphism.

Following [17], we study the map $f : \mathbb{T}^2 \to \mathbb{T}^2$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, defined through

$$f(x, y) = (x + y + g(x), y + g(x)) \bmod 1. \tag{1}$$

We will use the fundamental domain $[-\frac{1}{2}, \frac{1}{2})^2$ for the torus, so e.g., $x \bmod 1 = x - \lfloor x + \frac{1}{2} \rfloor$. If the "force" g were a degree-zero circle map, then (1) would be a generalized Chirikov standard map [1]. Instead, we assume that g is a degree-one, circle map:

$$g(x + 1) = g(x) + 1. \tag{2}$$

When g is a monotone increasing diffeomorphism, (1) is Anosov: every orbit is uniformly hyperbolic and f is topologically conjugate to Arnold's cat map $a : \mathbb{T}^2 \to \mathbb{T}^2$,

$$a(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1, \quad \text{where } A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{3}$$

More generally, Franks showed that (1) with (2) is semi-conjugate to a [9], i.e., there is a continuous, onto map $k : \mathbb{T}^2 \to \mathbb{T}^2$ such that

$$k \circ f = a \circ k. \tag{4}$$

The map k depends continuously on g, and when g is strictly monotone, k is a homeomorphism, implying – as mentioned above – that f is then conjugate a.

In [17], the first author made an attempt to elucidate the features of (1) when the circle map g acquires a critical fixed point,

$$g(x_p) = Dg(x_p) = 0. \tag{5}$$

In this case (1) has a parabolic fixed point $p = (x_p, 0)$ and is no longer Anosov. The main result of [17] was to show that the diffeomorphism acquires elliptic behavior when it crosses this Anosov boundary. Another feature of this map is the separation of the phase space into two regions, one in which the dynamics is nonhyperbolic and the other in which the diffeomorphism appears to be nonuniformly hyperbolic. Though neither of these statements were proved in [17], considerations in favor of these statements were presented.

In this paper we try to use numerical methods to substantiate the following assertions about (1) under the assumptions (2) and (5).

- There is an invariant, open region $E \subset \mathbb{T}^2$ whose boundary is formed from the stable and unstable manifolds of two fixed points of the map, a hyperbolic saddle, h, and a parabolic point, p. The Lebesgue measure of E is strictly less than that of $\mathbb{T}^2$.
- The channel E contains all non-hyperbolic orbits of f, and indeed has elliptic orbits for generic $\varepsilon > 0$.

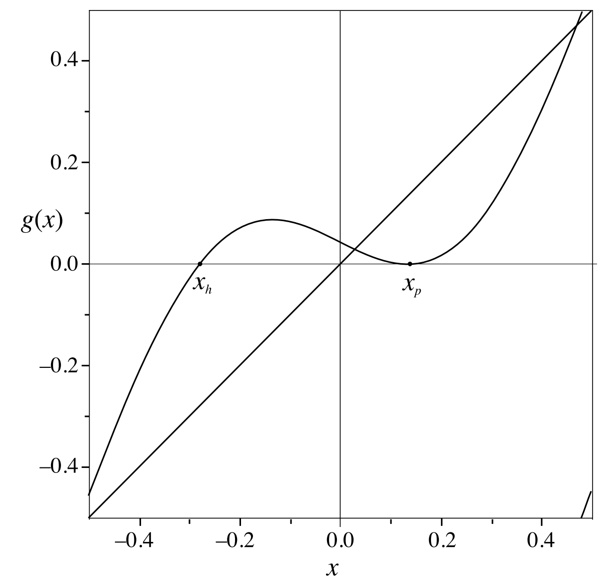

Fig. 1. Graph of the force g (6), for $\varepsilon = 0.5$ and $\mu = \mu_p(\varepsilon) \approx 0.27696$ from (9), with the parabolic point $x_p \approx 0.13386$ and saddle $x_h \approx -0.27889$.

- Conversely, the dynamics of $f|_H$, where $H = \mathbb{T}^2 \setminus E$, is nonuniformly hyperbolic; that is, the map is ergodic in H and has positive Lyapunov exponent.

Of course, these statements are purely numerical observations, which should therefore be considered mathematically as conjectures.

2. A parabolic standard map

Following [12,17], we study the dynamics of (1) using the degree-one circle map

$$g(x) = x + \frac{1}{2\pi} \left[\mu - (1 + \varepsilon) \sin(2\pi x)\right], \tag{6}$$

where μ and $\varepsilon \geq 0$, see Fig. 1. Note that when $\varepsilon = -1$ and $\mu = 0$, the map (1) reduces to Arnold's cat map (3).

The map f is a diffeomorphism whenever g is smooth. Indeed

$$f^{-1}(x, y) = (x - y, y - g(x - y)).$$

Moreover, this map is reversible, $f \circ S = S \circ f^{-1}$, with the "second" reversor of Chirikov's map (it does not have the first reversor since g is not odd when $\mu \neq 0$),

$$S(x, y) = (x - y, -y), \tag{7}$$

with the fixed set $\mathrm{Fix}(S) = \{(s, 0) : s \in \mathbb{T}\}$. Note that since S is an involution, the map

$$f \circ S(x, y) = S \circ f^{-1}(x, y) = (x - 2y + g(x - y), -y + g(x - y))$$

is also a reversor, with the fixed set

$$\mathrm{Fix}(f \circ S) = \left\{s + \tfrac{1}{2} g(s), \tfrac{1}{2} g(s) : s \in \mathbb{T}\right\}.$$

Under the assumption (5), $g(x) = \mathcal{O}((x - x_p)^2)$, the map (1) has a (symmetric) parabolic fixed point $p = (x_p, 0)$. For the case (6) this fixed point occurs at

$$x_p = \frac{1}{2\pi} \sec^{-1}(1 + \varepsilon) = \frac{1}{\pi}\sqrt{\frac{\varepsilon}{2}} \left(1 - \frac{5}{12}\varepsilon + \mathcal{O}(\varepsilon^2)\right), \tag{8}$$

when μ is chosen to be

$$\begin{aligned} \mu &= \mu_p(\varepsilon) \equiv (1 + \varepsilon) \sin(2\pi x_p) - 2\pi x_p \\ &= \sqrt{\varepsilon(2 + \varepsilon)} - \sec^{-1}(1 + \varepsilon) \\ &= \frac{\sqrt{8}}{3} \varepsilon^{3/2} \left(1 - \frac{9}{20}\varepsilon + \mathcal{O}(\varepsilon^2)\right). \end{aligned} \tag{9}$$