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Minimal topological chaos coexisting with a finite set of homoclinic and periodic orbits

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HIGHLIGHTS

- The pruning method can be applied to certain physical models.
- The combinatorics of the pruning map is found uncrossing invariant manifolds.
- Infinite pruning regions are related to singularities without rotation.

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ABSTRACT

In this note we explain how to find the minimal topological chaos relative to finite set of homoclinic and periodic orbits. The main tool is the pruning method, which is used for finding a hyperbolic map, obtained uncrossing pieces of the invariant manifolds, whose basic set contains all orbits forced by the finite set under consideration. Then we will show applications related to transport phenomena and to the problem of determining the orbits structure coexisting with a finite number of periodic orbits arising from the bouncing ball model.

1. Introduction

By minimal topological chaos relative to a homoclinic orbit P we mean the minimal structure of orbits that a system containing this homoclinic orbit can have in its isotopy class. It was Poincaré who realizes that the existence of such orbits implies a higher complexity [1], and Birkhoff and Smale proved that, under regular conditions, there are infinitely many periodic orbits in every neighbourhood of P [2–5].

It is known that a non-autonomous perturbation of an integrable system, satisfying Melnikov's conditions, creates homoclinic orbits with transversal intersection and also at least a chaotic set having a dense set of periodic orbits. See Fig. 1. Such models have many applications going from transport phenomena [6], the analysis of bifurcations in a driver oscillator [7] to the dynamics of bubbles in time-periodic straining flows [8]. In all these applications a natural question is the following: *which is the minimal periodic orbits structure that a map, having P as a homoclinic orbit, can have*? The same question can be formulated if P is a finite set of homoclinic and periodic orbits since chaotic behaviour can be created from the finite set of topological shapes induced by P. In [9] and references there in, periodic orbits are studied in applications to laser models, Lorentz and Rössler attractors, the Belousov–Zhabotinskii reaction, etc. To answer that question we need the notion of *forcing* introduced by P. Boyland.

Let f be a homeomorphism on the disk and let P be an orbit of f. The isotopy class of (P, f) is given by its braid type which identifies all the orbits that are equivalent to P under isotopies [10]. We say that (P, f) *forces* an orbit Q if every homeomorphism g isotopic to f relative to P, having an orbit with the braid type of P, must also has an orbit with the braid type of Q. The set of all the orbits whose braid types are forced by an orbit (P, f) will be denoted by Σ_P. Thus Σ_P contains a topological representative of each orbit that is forced by P, and it shows us the minimal description of the set of periodic orbits that a map can have given only a topological data.

One of the first result about the forcing relation of homoclinic orbits was stated by Handel in [11]. He provides conditions for

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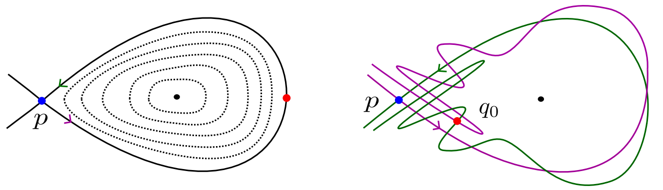

Fig. 1. Homoclinic orbit appearing after a non-autonomous perturbation of an integrable system.

ensuring that a finite set of homoclinic orbits imply the existence of a fixed point. In Hulme's thesis [12] there exists an extension of the Bestvina–Handel algorithm [13] which can be used for computing an efficient graph map or a generalized pseudoanosov representative within the isotopy class of a homoclinic orbit.

In [14–16] Collins has proposed a method for determining a graph representative whose orbits represent the dynamics forced by the homoclinic orbit P and, under certain conditions, construct a diffeomorphism that minimizes the topological entropy the isotopy class relative to P. This is done studying a trellis, a part of the homoclinic tangle of P. A similar motivation was given in [17] by Mitchell and Delos, where the attention was towards into the escape segments by iterations of the map.

All these methods can find exact or approximated symbolic dynamics in Σ_P but unfortunately the number of symbols is always increased as the trellis becomes more and more complicated and a computational cost is needed. Another disadvantage is that, except in a few cases, it is not clear how to apply them to the study of an infinitely many family of homoclinic orbits.

In [18] a pruning method is proposed for finding, given a homoclinic orbit, an Axiom A diffeomorphism whose non-wandering set realizes all the braid types forced by that orbit. This method can be considered as a differentiable version of the pruning theory developed by de Carvalho [19] for pruning surfaces homeomorphisms, and can be extended for finding Σ_P rel to a finite set of homoclinic and periodic orbits, since Σ_P is actually the *complement* of the pruning region rel to P. In [20] the technique was used for organizing certain horseshoe periodic orbits by forcing.

In fact, in this note we will explain how the pruning method works if P consists of certain infinite families of homoclinic orbits found in transport phenomena by Rom-Kedar in [21,22]. It will be showed, up isotopies, the pruning region rel to these orbits. Furthermore the method will be applied to a finite set of periodic orbits which include those ones studied by Tufillaro in [23] for the bouncing ball model, who has proposed a pruning region joining invariant manifolds. We improve his pruning region showing the existence of a map that realizes it which was not proved in [23]. We should note that the lines followed in this work can be adapted to a wide range of sets of periodic and homoclinic orbits arising from experimental data.

2. A model for minimal chaos

Our working model is the Smale horseshoe [5] which was one the first examples exhibiting deterministic chaos. This is a diffeomorphism F acting on a sub-disk of the disk as in Fig. 2. F is an Axiom A map, that is, F has hyperbolic structure on its non-wandering set which consists of an attractor point within the left semi-disk and a Cantor set K contained in the union of the rectangles $V_0 \cup V_1$. Then it was proved that F restricted to K is conjugated to the shift σ on the two-symbols compact space $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$. More general properties of Axiom A maps can be found in [24]. Collapsing segments joining two boundary points it is obtained the symbol square [25] represented in Fig. 2 as well.

We only devote our study to horseshoe homoclinic orbits of the form $q_0 = {}^{\infty}0.1w10^{\infty}$, where w is a finite word of symbols 0's and

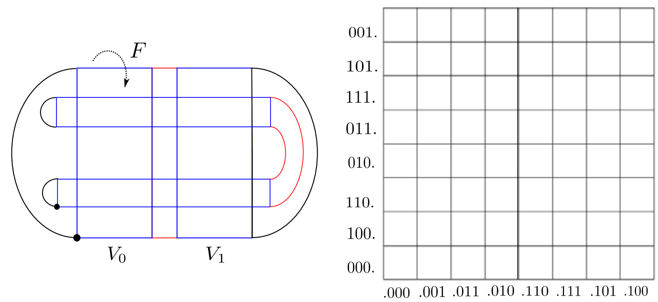

Fig. 2. The Smale horseshoe and its symbol square.

1's, that is, homoclinic orbits at the intersection of the stable and unstable manifolds of the fixed point with code 0^{∞}. These orbits often appear in dynamical applications in a wide range of systems as this one in Fig. 1.

Now we recall the pruning ideas proposed by Cvitanović in [26]. He has observed that certain dynamical systems are better understood if we consider them as incomplete or pruned horseshoes. This means that certain systems can be obtained from the uncrossing of pieces of the invariant manifolds of the Smale horseshoe or an another well-known Axiom A map. The regions where orbits were eliminated are called *pruning regions*. So the symbolic dynamics of the system corresponds to the symbolic dynamics of the horseshoe except the orbits included inside the pruning region. This powerful idea simplifies the orbit analysis since it is sufficient to find a good pruning region in order to describe the orbits structure.

Several authors as [25,27–30] have followed the pruning approach, and their results were directed to find rules for the remaining symbol dynamics, but no illumination was provided about how invariant manifolds influence the final grammar.

A pruning formalism was given in [19] by de Carvalho for pruning, in particular, the horseshoe F. It demands the existence of a pruning domain, that is, a topological simply connected domain D bounded by two segments θ_s and θ_u which belong to the stable manifold and the unstable manifold of periodic points, respectively. Then D is called a pruning domain if it satisfies the following condition:

$$F^n(\theta_s) \cap \mathrm{Int}(D) = \emptyset = F^{-n}(\theta_u) \cap \mathrm{Int}(D), \quad \forall n \geq 1. \tag{1}$$

Thus the pruning theorem [19] claims that condition (1) is sufficient for eliminating all orbit within $\mathrm{Int}(D)$ in the sense that an isotopy of F can be implemented in such a way that there are no recurrent points in $\mathrm{Int}(D)$ for the homeomorphism G at the end of the isotopy. As a consequence the non-trivial dynamics of G are given by σ on $\Sigma_2 \setminus \cup_{i\in\mathbb{Z}} F^i(\mathrm{Int}(D))$. Because this theorem reigns in the topological level in which there is not notion of invariant manifolds, this is not applicable to Cvitanović's pruning approach.

To solve that impasse one of us has proposed, in a joint work with A. de Carvalho [31], a differentiable version of the pruning theorem, that is used to prune Axiom A maps since hyperbolic structure allows us to make G, the end of the pruning isotopy, an Axiom A map too, although the most important property to point out is that this pruning isotopy uncrosses invariant manifolds in a controlled manner which means that uncrossings only happen in the interior of D and its iterates. See [18] for the details.

Recalling that a *bigon* $\mathcal{I}$ is a simply connected domain bounded by a segment of a stable manifold and a segment of an unstable manifold, it was proved in [18] that, given a homoclinic orbit P, Σ_P can be found eliminating all the bigons of F relative to P by successive prunings. Fig. 3 shows the elimination of a bigon $\mathcal{I}$ under the effect to the uncrossing of the invariant manifolds within D by a pruning isotopy.