AN IMPOSSIBILITY THEOREM FOR LINEAR SYMPLECTIC CIRCLE QUOTIENTS

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Using explicit computations of Hilbert series, we prove that when $d > 2$, a d-dimensional symplectic quotient at the zero level of a unitary circle representation V such that $V^{\mathbb{S}^1} = \{0\}$ cannot be $\mathbb{Z}$-graded regularly symplectomorphic to the quotient of a unitary representations of a finite group.

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1. Introduction

Let $G \to \mathrm{U}(V)$ be a unitary representation of a compact Lie group G on a finite-dimensional Hermitian vector space $(V, \langle\ ,\ \rangle)$. By convention, the Hermitian scalar product $\langle\ ,\ \rangle$ is assumed to be complex antilinear in the first argument. For the infinitesimal action $d/dt \exp(-t\xi).v_{|t=0}$ of an element ξ of the Lie algebra $\mathfrak{g}$ of G on an element $v \in V$ we write $\xi.v$. The *moment map* of the representation is the regular map

$$J : V \to \mathfrak{g}^*, \qquad v \mapsto J_\xi(v) := (J(v), \xi) := \frac{\sqrt{-1}}{2}\langle v, \xi.v\rangle,$$

where $(J(v), \xi)$ stands for the dual pairing of $J(v) \in \mathfrak{g}^*$ with an arbitrary $\xi \in \mathfrak{g}$. Since J is an equivariant map, its zero level $Z := J^{-1}(0)$ is G-saturated, and one can define the *symplectic quotient* $M_0 := Z/G$. If G is finite, our convention is to set $Z := V$ and refer to the quotient as a *linear symplectic orbifold*. Note that when G is not discrete, $0 \in \mathfrak{g}^*$ is a singular value of J, and $Z \subset V$ is a closed cone. In our situation, the symplectic quotient M_0 is not a manifold, but a semi-algebraic set stratified by symplectic manifolds of varying dimensions [30]. We regard it as a *Poisson differential space* with a *global chart* provided by a Hilbert basis (cf. [10]). When comparing two such spaces, there is a natural notion of equivalence, namely that of a *$\mathbb{Z}$-graded regular symplectomorphism*. This point of view will be explained in more detail in Subsection 2.1, see also [10].

The objective of this paper is to use the computations of the Hilbert series in [17] to compare symplectic quotients of unitary circle representations with finite unitary quotients. Our main result is the following

THEOREM 1. *Let $\mathbb{S}^1 \to \mathrm{U}(V)$ be a unitary circle representation such that $V^{\mathbb{S}^1} = \{0\}$. If the dimension of the symplectic quotient M_0 is $d > 2$, then there cannot exist a $\mathbb{Z}$-graded regular symplectomorphism of M_0 to a quotient of $\mathbb{C}^n$ by a finite subgroup $\Gamma < \mathrm{U}_n := \mathrm{U}(\mathbb{C}^n)$.*

Note we can assume $n = d/2$. In [10], one can find a constructive proof of the statement that any 2-dimensional symplectic quotient of a unitary torus representation is $\mathbb{Z}$-graded regularly symplectomorphic to some cyclic quotient of $\mathbb{C}$. We can always assume without loss of generality that $V^{\mathbb{S}^1} = \{0\}$, because $V^{\mathbb{S}^1}$ is a symplectic subspace on which the circle acts trivially; see Remark 5. So together with [10] and the observation that the 0-dimensional case is trivial, Theorem 1 provides the complete answer to the question of which linear symplectic circle quotients are $\mathbb{Z}$-graded regularly symplectomorphic to a finite unitary quotient.

In [16] with coauthor G. W. Schwarz, we use invariant theory and the topological properties of the associated GIT quotients to prove a stronger statement that does not involve the grading of the ring of regular functions; see [16, Theorem 1.5]. The point of this paper is that Theorem 1 can be obtained using the properties of restrictions of $\mathbb{Z}$-graded regular symplectomorphisms established in Section 3 along with some brute force computations involving the Hilbert series.

It should be mentioned that some classes of unitary circle representations can be ruled out from our considerations right from the beginning. If M_0 is homeomorphic to a finite symplectic quotient, it is necessarily a rational homology manifold. Assuming $V^{\mathbb{S}^1} = \{0\}$, this is the case precisely when the sign of exactly one weight of the representation differs from the sign of the others, see [14] or [10, Theorems 3 and 4]. In this situation, the corresponding GIT quotient $V/\!\!/\mathbb{C}^\times$, which is homeomorphic to M_0 by the Kempf–Ness theorem [19, 28], is an orbifold (cf. [33]). It is perfectly legitimate to restrict our attention to these cases. Moreover, in [17] the authors discovered a Diophantine condition on the weights (cf. Eq. (2.7)) that has to hold if the symplectic circle quotient is $\mathbb{Z}$-graded regularly symplectomorphic to a finite unitary quotient. Example calculations ([17, Section 7]) indicate that the 'majority' of unitary circle representations violate this condition, even though it holds in infinitely many cases. The result of this paper is that the remaining cases can be ruled out as well.

Let us outline the plan of the paper. In Section 2.1, we recall the basics from [10] about how to construct $\mathbb{Z}$-graded regular symplectomorphism from global charts