Vol. 75 (2015) *REPORTS ON MATHEMATICAL PHYSICS* No. 3

SYMMETRIES IN LAGRANGIAN FIELD THEORY

LUCIA BÚA

Departamento de Xeometría e Topoloxía, Facultad de Matemáticas, USC, 15782-Santiago, Spain
(e-mail: lucia.bua@usc.es)

IOAN BUCATARU

Faculty of Mathematics, Alexandru Ioan Cuza University, Iaşi, Romania
(e-mail: bucataru@uaic.ro)

MANUEL DE LEÓN

ICMAT CSIC, Madrid, Spain
(e-mail: mdeleon@icmat.es)

MODESTO SALGADO

Departamento de Xeometría e Topoloxía Facultad de Matemáticas, USC, 15782-Santiago, Spain
(e-mail: modesto.salgado@usc.es)

and

SILVIA VILARIÑO

Centro Universitario de la Defensa de Zaragoza & I.U.M.A. Academia General Militar.
Carretera de Huesca s/n. 50090 Zaragoza, Spain
(e-mail: silviavf@unizar.es)

(*Received August 25, 2014 – Revised January 7, 2015*)

By generalising the cosymplectic setting for time-dependent Lagrangian mechanics, we propose a geometric framework for the Lagrangian formulation of classical field theories with a Lagrangian depending on the independent variables. For that purpose we consider the first-order jet bundles $J^1\pi$ of a fiber bundle $\pi : E \to \mathbb{R}^k$ where $\mathbb{R}^k$ is the space of independent variables. Generalized symmetries of the Lagrangian are introduced and the corresponding Noether theorem is proved.

Keywords: symmetries, Cartan theorem, Noether theorem, conservation laws, jet bundles.

1. Introduction

As it is well known, the natural arena for studying mechanics is symplectic geometry. One interesting problem is to extend this geometric framework for the case of classical field theories. Several different geometric approaches are well known: the polysymplectic formalisms developed by Sardanashvily et al. [11, 12, 35] and by Kanatchikov [17], as well as the n-symplectic formalism of Norris [27, 29] and the k-cosymplectic of de León et al. [22, 23].

Let us remark that the multisymplectic formalism is the most ambitious program for developing the classical field theory (see for example [2, 9, 10, 13, 14, 18], and references quoted therein).

The aims of this paper are:

- To give a new Lagrangian description of first-order classical field theory, by considering a fibration $E \to \mathbb{R}^k$, which has as particular cases the cosymplectic setting for time-dependent Lagrangian mechanics, and it is related with the k-cosymplectic [23] and multisymplectic formalisms. Let us observe that although every fiber bundle over $\mathbb{R}^k$ is a trivial bundle since $\mathbb{R}^k$ is contractible (Steenrod 1951 [36]), we do not use this fact to develop our Lagrangian description.
- To introduce and to study the generalized symmetries in first-order Lagrangian field theory. For time-dependent Lagrangian mechanics this was done by J. F. Cariñena et al. [5].

In the present paper we present a new approach for Lagrangian field theory, working with the first-order jet bundle $J^1\pi$ of a fiber bundle $\pi : E \to \mathbb{R}^k$, where E is $(n+k)$-dimensional.

The crucial point is that each 1-form on $\mathbb{R}^k$ defines a tensor field of type (1, 1) on $J^1\pi$, see Saunders [38].

The paper is organized as follows. The main tools to be used are those of vector fields, k-vector fields and forms along maps, the general definitions of which are given in Section 2.

In Section 3 we introduce the geometric elements on $J^1\pi$ necessary to develop the geometric formulation of the Euler–Lagrange field equations in Section 4 and to the study of symmetries and conservations laws. The principal tools, here described, are the canonical vector fields, the k vertical endomorphisms and a kind of k-vector fields, known as SOPDEs, which describe systems of second-order partial differential equations.

This machinery is later used to discuss symmetries in this context, extending some previous results (see [1, 26]).

The geometric formulation of the Euler–Lagrange field equations is given in Section 4, see Theorem 1. For this purpose we introduce the k Poincaré–Cartan 1-forms using the Lagrangian and the k vertical endomorphisms.

Our formulation is a natural extension of the k-cosymplectic formalism developed in [23] as we show in Section 4.3.

Section 5 is devoted to discussing symmetries and conservation laws. We introduce symmetries of the Lagrangian and we give a Noether theorem.

2. Preliminaries

2.1. k-vector fields

A system of first-order ordinary differential equations, on a manifold M, can be geometrically described as a vector field on M. Accordingly, a system of first-order