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PERTURBATIVE SEMICLASSICAL TRACE FORMULAE FOR HARMONIC OSCILLATORS

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In this article we extend previous semiclassical studies by including more general perturbative potentials of the harmonic oscillator in arbitrary spatial dimensions. Our starting point is a radial harmonic potential with an arbitrary even monomial perturbation, which we use to study the resulting $\mathrm{U}(D)$ to $\mathrm{O}(D)$ symmetry breaking. We derive the gross structure of the semiclassical spectrum from periodic orbit theory, in the form of a perturbative ($\hbar \to 0$) trace formula. We then show how to apply the results to even-order polynomial potentials, possibly including mean-field terms. We have drawn the conclusion that the gross structure of the quantum spectrum is determined from only classical circular and diameter orbits for this class of systems.

Keywords: perturbative trace formula, semiclassical density of states, radially perturbed harmonic oscillators.

1. Introduction

In 1913 Niels Bohr published his seminal work on the hydrogen atom [1] where he depicted the electron orbiting the proton as planets orbit the sun. Bohr's pictorial model is used in logotypes of research institutions and companies world wide, and is still the most popular way to draw an atom. With the vocabulary of today, Bohr obtained the quantum mechanical energy levels of the Coulomb potential. But 100 years ago, there was hardly any established quantum theory. Bohr's way of combining well-known classical mechanical laws with an innovative quantization of the electron radius (or equivalently, its angular momenta), together with the correspondence principle, had a tremendous influence on the development taking place in the following two decades. Bohr only included circular orbits, although the corresponding classical system have elliptic solutions [2]. It must be considered of great historical importance for the development of quantum mechanics that Bohr obtained the correct quantum spectrum from his simple model. At first sight, the development of the Heisenberg–Schrödinger quantum theory seemed to be unrelated to Bohr's semiclassical treatment. However, works by Einstein [3], Brillouin [4] and Keller [5] (EBK), Van Vleck [6], and Feynman [7], have pointed onto relations between the action for the classical orbits of a particle and the corresponding quantum spectrum. The interest in relations between classical systems and their quantum counterpart boosted again in the 60s and 70s, due to new powerful computers, with the study of quantum chaos [8–10]. Finally a so called periodic orbit theory (POT) was introduced for chaotic systems by Gutzwiller [11], and for different regular systems by Balian and Bloch [12] and Berry, Mount and Tabor [13, 14], and others. Now the studies of the periodic classical orbits were related to quantum mechanical observables through so-called trace formulae (TF), which had already been studied in the 50s by Selberg [15]. A trace formula expresses the spectrum of a differential operator, as for example in the time independent Schrödinger equation, represented by a train of delta functions, with a sum over the classical periodic orbits. Several principally important quantum systems: the harmonic oscillator; the cavity; and again, the hydrogen atom, were soon analyzed within POT [12, 16]. These systems could all be connected by the principle trace formula for integrable systems given by Berry and Tabor [14]. At the same time, semiclassical approximations were successful in describing shell structures of different quantum many-body systems from atomic and nuclear physics [17, 18]. This was one motivation to also refine the POT further to classical chaotic systems. Studies of (super-) shell structures have now been undertaken in new man-made systems: the abundance in atomic metal clusters [19], that have been confirmed experimentally [20]; in solid state devices, like the conductance of quantum wires [21, 22]; in weakly repulsive atomic Fermi gasses [23], and pairing gaps of attractive Fermi gasses [24], nano-grains [25] and recently also in triangular flakes of graphene [26].

In this article we treat a class of quantum systems of principal importance in approximations, the isotropic perturbed harmonic oscillator (HO) in arbitrary dimensions. We here present a TF for $\mathrm{U}(D)$ to $\mathrm{O}(D)$ symmetry breaking that gives the gross structure of density of states (DOS), sometimes called the level density, to leading order in $\hbar^{-1}$ for the perturbed system and that recovers the quantum mechanical TF in the limit of no perturbation to leading order in $\hbar^{-1}$. The breaking of $\mathrm{U}(D)$ symmetry for collections of HOs can have future relevance for many different applications, from nuclear physics and clusters to more recent systems with cold quantum gasses and graphene. Special cases have been presented before: the quartically perturbed two-dimensional HO was treated by Creagh in [27]; and the three-dimensional counterpart by Brack *et al.* in [28]. The present treatment follows a similar perturbative technique as pioneered by Creagh [27], but generalizes the special case of quartic perturbation and also allows the treatment of arbitrary dimensions simultaneously.

2. The D-dimensional harmonic oscillator

We consider the Hamiltonian of the D-dimensional harmonic oscillator (HO), as given by the following Hamiltonian function defined from the classical space and