ON THE METRIC OPERATOR FOR A NONSOLVABLE NON-HERMITIAN MODEL

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We study a non-Hermitian Hamiltonian the eigenvalues and eigenfunctions of which are not known explicitly. For this Hamiltonian we derive a closed formula for the metric operator and compute the Hermitian counterpart of the non-Hermitian Hamiltonian.

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1. Introduction

In this paper, the terms "Hermitian" and "self-adjoint" as well as the terms "non-Hermitian" and "non-self-adjoint" are used as synonyms.

In quantum mechanics, the observables (Hilbert space operators associated with the physical observables) and in particular the Hamiltonian (Schrödinger operator associated with the energy of the quantum-mechanical system) must be Hermitian because Hermiticity guarantees that the energy spectrum is real and that time evolution is unitary (probability-preserving). On the other hand, there are many problems in physics described by non-Hermitian operators [1] and there are non-Hermitian Hamiltonians in quantum mechanics describing open systems [2]. However, in the present paper main motivations for our work are recent attempts to develop the so-called quasi-Hermitian quantum mechanics, where physical observables are represented by quasi-Hermitian operators. Quasi-Hermitian operators form an important class of operators with real spectrum and they need not be Hermitian. In quasi-Hermitian quantum mechanics the main feature is that the non-Hermitian quasi-Hermitian operator can represent the full quantum system (closed quantum system) so that the quasi-Hermitian quantum mechanics is just the conventional quantum mechanics with possible representation of physical observables by quasi-Hermitian operators not necessarily by Hermitian operators.

A densely defined (unbounded, in general) operator $H\colon D(H) \subset \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called *quasi-Hermitian* if there exists a bounded, positive (i.e.

$\langle \Theta f, f \rangle > 0$ for all $f \in \mathcal{H}$, $f \neq 0$), and boundedly invertible operator Θ, called a *metric operator*, such that

$$H^*\Theta = \Theta H \qquad (\text{i.e., } H^* = \Theta H \Theta^{-1}), \tag{1}$$

where H^* denotes the adjoint operator of H in the Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$. It is obvious that if we define a new inner product $\langle \cdot, \cdot \rangle_+ = \langle \Theta \cdot, \cdot \rangle$ then H becomes Hermitian with respect to $\langle \cdot, \cdot \rangle_+$ and therefore any quasi-Hermitian operator has a real spectrum.

Mostafazadeh observed in [3, 4] that if the operator Θ decompose to

$$\Theta = \Omega^*\Omega,$$

where Ω is a bounded and boundedly invertible operator (one example of such Ω is obviously $\sqrt{\Theta}$, the unique positive square root of Θ), then the operator

$$h = \Omega H \Omega^{-1}$$

is Hermitian with respect to $\langle \cdot, \cdot \rangle$. The operator h is called a *Hermitian counterpart* of the non-Hermitian operator H. In such a case, the operator H is called a non-Hermitian (quasi-Hermitian) representation of the Hermitian operator (observable) h.

The quasi-Hermitian quantum mechanics was introduced by Scholtz, Geyer and Hahne in [5] and developed further and made popular by Bender, Znojil, Mostafazadeh and other authors (see [6–8] and references given therein).

In some cases, a non-Hermitian representation of the given observable may be more convenient for manipulations. This motivates to investigate the problem whether a given non-Hermitian operator with a real spectrum may be a quasi-Hermitian representation of an observable. In other words, for a given non-Hermitian operator having real spectrum one attempts to find a metric operator for it, if exists, and compute a Hermitian counterpart of the non-Hermitian operator. These problems turned out in general to be rather difficult mathematical problems. So far, one has succeeded to compute exact expressions for the metric operators and Hermitian counterparts in very few cases of non-Hermitian Hamiltonians which are solvable models (their eigenvalues and eigenfunctions are calculated explicitly); for some of them see [9–16]. Note that in [15, Section 4.5] the authors have also given an example for which the metric operator is found explicitly even if the spectral problem is not explicitly solvable.

In the present paper, we deal with the nonsolvable explicitly eigenvalue problem

$$-\psi''(x) + q(x)\psi(x) = \lambda\psi(x), \qquad x \in (-1, 0) \cup (0, 1), \tag{2}$$

$$\psi(0^-) = \alpha\psi(0^+), \qquad \psi'(0^-) = \alpha\psi'(0^+), \tag{3}$$

$$\psi(-1)\cos A + \psi'(-1)\sin A = 0, \qquad \psi(1)\cos B + \psi'(1)\sin B = 0, \tag{4}$$

where $q(x)$ is a real-valued piecewise continuous function, λ is a spectral parameter, α is a fixed nonzero complex number, A and B are given real numbers, and ψ is a desired solution from the Sobolev space $W^{2,2}((-1, 0) \cup (0, 1))$. The space $W^{2,2}((-1, 0) \cup (0, 1))$ consists of all complex-valued functions $\psi \in L^2(-1, 1)$