GROUP ANALYSIS AND NONLINEAR SELF-ADJOINTNESS FOR A GENERALIZED BREAKING SOLITON EQUATION

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(*Received November 6, 2013*)

In this paper, a variable coefficient $(2+1)$-dimensional generalized breaking soliton equation is considered by means of the Lie group method. Having written an equation as a system of two dependent variables, we perform a complete group classification for the system. Consequently, for arbitrary functions, solutions of the generalized breaking soliton equation are connected with the ones of a variable coefficient Korteweg–de Vries equation by a transformation. For other cases, the reduced equations and exact solutions are constructed. Meanwhile, we prove that the system is nonlinearly self-adjoint and construct the general conservation law formulae.

Keywords: Lie group method, nonlinear self-adjointness, conservation law, generalized breaking soliton equation.

1. Introduction

It is well known that modeling the phenomena in nature by partial differential equations (PDEs) is one of the central problems of mathematical physics and applied mathematics and thus attracts attention of researchers in the associated fields. In order to obtain more accurate information about the models, some PDEs contain some arbitrary parameters or functions which are not fixed and later must be determined by practical applications, also some PDEs have been extended to higher dimensions [1]. For instance, the Kadomtsev–Petviashvili equation, which has two-periodic wave solutions characterizing the shallow water waves with more accuracy, is a $(2+1)$-dimensional extension of the celebrated Korteweg–de Vries (KdV) equation [2].

The problem of study of higher-dimensional PDEs involving arbitrary parameters or functions, such as constructing exact solutions and conserwation laws, triggered more new methods. The theory of Lie groups and Lie algebras, introduced by Sophus Lie (1842–1899) and also named symmetry group, evolved into one of the most important development of mathematics and physics and exerted important effects in diverse fields [3, 4]. For example, the symmetry group admitted by PDEs can transform higher-dimensional PDEs to the familiar lower-dimensional ones and then allows to obtain exact solutions of the original PDEs. It can also help to determine arbitrary elements in the PDEs to search for further information about the models.

Symmetry also plays an important role in constructing conservation laws of PDEs. When PDEs admit a variational structure, Noether theorem gives the general conservation law formula by means of variational symmetries. If PDEs are not obtained from a variational principle, some new methods are developed to achieve the goal [5–7]. Recently, the concept of nonlinear self-adjointness [7] including the subclasses stated earlier [8, 9], provided a feasible method to construct conservation laws of PDEs. The main idea of the method, which traces back to the literature [10, 11] and references therein, is to turn the system of PDEs into Lagrangian equations by artificially adding additional variables, then to apply the theorem proved in [12] to construct conservation laws. Approximate nonlinear self-adjointness and approximate conservation laws were considered in [13].

Since many useful models in theoretical and applied sciences admit rich symmetry structures that follow from physical laws, such as e.g. from Galilean or relativistic theories, thus another problem is to determine which differential equation selecting from a broad class of possible PDEs is the best model to reflect the natural laws from the point of view of the Lie group theory [14]. The answer to this question is the group classification. The main idea of group classification is to classify the arbitrary parameters or functions contained in PDEs which make that PDEs admit more symmetry groups. Generally speaking, to find complete group classifications of such equations in terms of their unknown parameters is a complicated problem that challenges researchers. The principal tool for handling group classification is the classical infinitesimal routine developed by Sophus Lie. It transforms the problem to finding the corresponding Lie symmetry algebra of infinitesimal operators whose coefficients are found as solutions of some over-determined system of linear PDEs [3, 4].

Here, we will pay attention to a variable-coefficient (2+1)-dimensional generalized breaking soliton equation of the form [15]

$$u_t + a(t)\,u_{xxx} + 6\,a(t)\,u\,u_x + b(t)\,u_{xxy} + 4\,b(t)\left(u\int u_y\,dx\right)_x = 0, \tag{1}$$

where x and y are the scaled space coordinates, t is the scaled time coordinate, u is a function of x, y and t, and $a(t)$ and $b(t)$ are analytical functions of t. Eq. (1) reduces to the variable coefficient KdV equation when $y = x$ or $b(t) = 0$. Bilinear forms and N-soliton solutions of Eq. (1) are obtained in [16]. For Eq. (1), we assume that at least one of $a(t)$ and $b(t) \neq 0$ is a nonconstant function because the constant case had been studied in [17–22].

Historically, there are a number of papers contributing to the studies of subclasses of Eq. (1). For example, a 'typical' $(2+1)$-dimensional breaking soliton equation

$$u_t + \omega\,u_{xxy} + 4\,\omega\,(u\int u_y\,dx)_x = 0 \tag{2}$$