Physica D 313 (2015) 11–25

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Singularity confinement and full-deautonomisation: A discrete integrability criterion

B. Grammaticos [a,*], A. Ramani [b], R. Willox [c], T. Mase [c], J. Satsuma [d]

[a] IMNC, Université Paris VII & XI, CNRS, UMR 8165, Bât. 440, 91406 Orsay, France

[b] Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau, France

[c] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, 153-8914 Tokyo, Japan

[d] Department of Mathematical Engineering, Musashino University, 3-3-3 Ariake, Koto-ku, 135-8181 Tokyo, Japan

HIGHLIGHTS

- An efficient discrete integrability detector is presented.
- Paradoxes related to singularity confinement are resolved.
- The approach is justified by a rigorous, algebro-geometric, analysis.
- The notion of early, standard and late confinement are introduced.
- Late confinement helps resolve a paradox related to gauge freedom.

ARTICLE INFO

Article history:
Received 11 May 2015
Received in revised form
5 August 2015
Accepted 7 September 2015
Available online 14 September 2015
Communicated by J. Bronski

Keywords:
Mapping
Integrability
Deautonomisation
Singularity
Degree growth
Algebraic entropy

ABSTRACT

We present a new approach to singularity confinement which makes it an efficient and reliable discrete integrability detector. Our method is based on the full-deautonomisation procedure, which consists in analysing non-autonomous extensions of a given discrete system obtained by adding terms that are initially absent, but whose presence does not alter the singularity pattern. A justification for this approach is given through an algebro-geometric analysis. We also introduce the notions of early and late confinement. While the former is a confinement that may exist already for the autonomous system, the latter corresponds to a singularity pattern longer than that of the autonomous case. Late confinement will be shown to play an important role in the singularity analysis of systems with non-trivial gauge freedom, for which the existence of an undetected gauge in conjunction with a sketchy analysis, might lead to erroneous conclusions as to their integrability. An algebro-geometric analysis of the role of late confinement in this context is also offered. This novel type of singularity confinement analysis will be shown to allow for the exact calculation of the algebraic entropy of a given mapping.

1. Introduction

Singularity confinement [1] is a discrete analogue of the Painlevé property [2] of ordinary differential equations, which infers the integrability of a given equation from the local structure of its singularities. The crucial requirement there is that singularities, the position of which depends upon the initial conditions, do not introduce multivaluedness (which in general makes it impossible to represent the solution of the differential equation as a function). Analogously, the singularity confinement approach is based on the local study of the singularities that appear in a discrete system. Here as well we are interested in singularities with positions that depend on the initial conditions of the system and singularity confinement requires those singularities to disappear after a few iteration steps, lest they lead to indeterminacies that make the construction of the solution of the system impossible. The relevance of singularity confinement as an integrability detector is strengthened by the fact that all discrete systems integrable through spectral methods, studied to date, have been shown to possess confined singularities. On the other hand, linearisable discrete systems in general do not satisfy the singularity confinement criterion [3], in close parallel to what happens in the continuous case, where linearisable differential systems in general do not possess the Painlevé property either [4].

* Corresponding author. Tel.: +33 169157201; fax: +33 169157196.
E-mail address: grammaticos@univ-paris-diderot.fr (B. Grammaticos).

http://dx.doi.org/10.1016/j.physd.2015.09.006
0167-2789/

If this parallel between singularity confinement and the Painlevé property had been perfect, we would of course have been in possession of an efficient and convenient discrete integrability detector. However, the discovery of non-integrable systems with confined singularities called the usefulness of singularity confinement as an integrability criterion into question. The best-known example of such a mapping is the one proposed by Hietarinta and Viallet in [5], which we refer to as the H–V mapping:

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n^2}. \tag{1}$$

The pattern of its singularities is $\{x_{n+1} = 0, x_{n+2} = \infty, x_{n+3} = \infty, x_{n+4} = 0\}$ and, since $x_{n+5} = x_n$, its singularity is confined. However the authors of [5] have shown numerically that this mapping exhibits large scale chaos and thus cannot be expected to be integrable. This of course raises the question what singularity confinement might mean in this case? Clearly, the confinement property is related to some subtle cancellations occurring when one iterates a rational mapping. These cancellations will in fact reduce the growth of the degree of the successive iterates. (We should point out here that, as shown by Bellon and Viallet [6], while the degree itself is not invariant under coordinate changes its growth is invariant and thus characteristic for the mapping). In fact, when a mapping is integrable, these confinement-related cancellations slow down the degree growth to such an extent that, asymptotically, it becomes polynomial [7]. However, in the case of the H–V mapping, whereas some cancellations do take place, these do not suffice to curb the asymptotic degree growth which remains exponential. Such rapid growth is the signature of non-integrability. Hietarinta and Viallet [5] therefore introduced a quantitative measure of the degree growth of a rational mapping: its algebraic entropy. If d_n represents the homogeneous degree of the numerator or denominator of x_n, the algebraic entropy of the mapping is given by the limit $\mathcal{E} = \lim_{n\to\infty} \frac{1}{n} \log d_n$. For an integrable mapping the algebraic entropy must vanish. On the other hand, a non-zero value for $\mathcal{E}$ implies exponential growth and is therefore an indication of non-integrability. In the case of mapping (1) the algebraic entropy can be computed exactly [8] and is found to be $\mathcal{E} = \log(\frac{3+\sqrt{5}}{2})$.

Curiously, this mapping remained essentially a singleton as far as counterexamples to the confinement criterion were concerned (but see [9]), despite the fact that the authors of [5] presented general arguments for the existence of whole families of confining, non-integrable, mappings. The status of singularity confinement became even more complicated with the issue of late confinement [10]. While standard practice in the implementation of singularity confinement, for mappings with parametric freedom, had been to enforce confinement at the very first possibility, it was not at all clear at the time why one should abide by this rule and why, for example, one could not postpone confinement until a later occasion. It turns out [10,11] that when one opts for a late confinement, the resulting system will be non-integrable despite its singularities being confined (and despite the fact that when confinement is implemented normally, the resulting system might be integrable).

These problems led to a certain distrust of singularity confinement as a method for detecting or deriving discrete integrable systems. Still, there has always existed a domain – for which we coined the term deautonomisation [12] – where this criterion continued to thrive and in fact furnished a slew of novel results. What we mean by *deautonomisation*, is to consider the free parameters of a mapping (which a priori take constant values) to be functions of the independent variable, the precise form of which has to be obtained through the use of a certain discrete integrability criterion. The rationale behind this approach lies in the relation the growth properties of the solution of a mapping bear to its integrability. In the deautonomisation procedure one starts from an integrable autonomous system, obeying the low-growth requirement, and one seeks to extend it to a non-autonomous form while keeping the same growth. In most practical applications however, the integrability criterion one uses is in fact singularity confinement. The reason being that, compared to techniques that rely on the calculation of the algebraic entropy, the confinement criterion has the immense advantage that one can examine each singularity separately, establishing the constraints on the parameters one at a time and not all at once in a hopelessly entangled way.

It is precisely this very same deautonomisation approach that will be shown to reinstate singularity confinement as a reliable discrete integrability criterion. In [13] we introduced the so-called *full-deautonomisation* approach, and we claimed that this is the proper way to perform the singularity analysis of a given mapping. If the system is integrable, the characteristic equations for the constraints that one obtains for the parameters, will only have roots with modulus 1, whereas the presence of a root with modulus greater than 1 implies non-integrability. In what follows we shall first illustrate the power of this approach through several examples, after which shall give a detailed discussion of the problems that arise due to gauge freedom in the mapping and of the solution the concept of late confinement offers to this conundrum.

2. The full-deautonomisation procedure

The deautonomisation procedure consists in assuming that the parameters that appear in a mapping are functions of the independent variable and in using some integrability criterion, like singularity confinement, to fix their precise form. Standard practice when applying this procedure is to require that the (confined) singularity pattern of the autonomous mapping and that of its non-autonomous extension be identical. (An analogous requirement can be formulated whenever the algebraic entropy criterion is used: one then requires that the degree growths are the same for the autonomous and non-autonomous mappings). The full-deautonomisation procedure is an extension of the standard one, in which one not only lets the parameters in the mapping be functions of the independent variable, but in which one also introduces terms (with non-autonomous coefficients) that do not appear in the original mapping, as long as such terms do not alter the singularity pattern of the original mapping.

We shall illustrate this procedure and its implications for the integrability of a given mapping on two examples. The first one is a mapping of the form

$$x_{n+1} + x_{n-1} = \frac{1}{x_n^2}. \tag{2}$$

Its singularity pattern is $\{0, \infty^2, 0\}$, where by ∞^2 we mean that if we introduce a small quantity ϵ and assume that x_n is finite and $x_{n+1} = \epsilon$, then x_{n+2} will be of order $1/\epsilon^2$. Deautonomising (2) then consists in replacing the numerator of the right-hand side by a function a_n and to require the mapping to have confined singularities with exactly the same pattern as the autonomous one. This yields the constraint $a_{n+1} = a_{n-1}$, which gives an integrable, but trivial, non-autonomous extension of (2). In fact, by introducing the appropriate gauge $x_n \to \gamma_n x_n$, with $\gamma_n^3 = a_n^2/a_{n-1}$ and $\gamma_{n+1} = \gamma_{n-1}$, we can put $a_n = 1$ for all n.

In order to proceed to the full-deautonomisation of (2) we must add terms that do not modify the initial singularity pattern. It is straightforward to convince oneself that the only possible such extension is by adding a term inversely proportional to x, which leads to

$$x_{n+1} + x_{n-1} = \frac{b_n}{x_n} + \frac{a_n}{x_n^2}. \tag{3}$$