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Asymptotic analysis of a viscous thread extending under gravity

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HIGHLIGHTS

- Solve full initial–boundary-value problem for thread falling under gravity.
- Determine how, eventually, the inertial terms must become important.
- Determine criteria such that surface-tension-driven pinching will not occur.

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ABSTRACT

Despite extensive research on extensional flows, there is no complete explanation for why highly viscous fluids extending under gravity can form such persistent and stable filaments with no sign of destabilization from surface tension. We therefore investigate the motion of a slender axisymmetric viscous thread that is supported at its top by a fixed horizontal surface and extends downward under gravity. In the case in which inertia and surface tension are *initially* negligible, we consider the long-wavelength equations for the full initial–boundary-value problem for a thread of arbitrary initial shape. We show that, eventually, the accelerations in the thread become sufficiently large that the inertial terms become important. Thus, we keep the inertial terms and, using matched asymptotic expansions, obtain solutions for the full initial–boundary-value problem. We show that the dynamics can be divided into two generic cases that exhibit very different behaviour. In the first case, the thread develops a long thin region that joins together two fluid masses. In this case, we use order-of-magnitude estimates to show that surface-tension-driven pinching will not occur if the square root of the Reynolds number is much greater than the initial aspect ratio divided by the Bond number. In the second case, the thread becomes thin near the horizontal surface. In this case, we show that the long-wavelength equations will ultimately break down and discuss the role of inertia in determining the dynamics. The asymptotic procedures require a number of novel techniques and the resulting solutions exhibit surprisingly rich behaviour. The solution allows us to understand the mechanisms that underlie highly persistent filaments.

1. Introduction

The analysis of flows of filaments represents one of the most important problems in classical fluid mechanics. Work on this subject dates back over 100 years to the pioneers of fluid mechanics such as Rayleigh [1] and Trouton [2]. A particularly natural problem is that of a thread of highly viscous fluid attached to a horizontal surface from below and extending under gravity. Although this problem has received significant attention [3–9], there are still a number of outstanding fundamental questions about such flows. Some of these questions are highlighted in a recent review paper by Eggers & Villermaux [10]. They describe the substantial progress and the sometimes remarkable agreement between theory and experiments for surface-tension-driven pinching of thin filaments at high and moderate Reynolds numbers. However, in a subsection entitled ‘Honey, why are you so thin?’, they discuss experiments in which very viscous threads extending under gravity form extraordinarily long filaments with no sign of destabilization from surface tension. A similar phenomenon can occur in the fabrication of glass microelectrodes [6,7]. Here, a long thin glass tube is hung

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from a support and a weight is attached to the bottom. The glass is then heated so that the viscosity of the glass decreases and relatively rapid extension occurs. If fracturing of the glass does not occur (e.g. if the glass temperature is sufficiently high), this process can also lead to the formation of extremely long thin filaments.

Indeed, the question of whether these threads can extend indefinitely in the absence of noise remains an open question. In this paper, we will address the need for a quantitative theory by developing an asymptotic theory that can solve the full initial–boundary-value problem for a thread with arbitrary initial shape.

We now briefly review the previous work related to the current problem. A number of authors have considered the problem in which inertia is completely neglected. Wilson [3] used a long-wavelength theory to consider the extrusion of a viscous fluid from a narrow vertical tube in the presence of gravity. His theory, that neglects inertia, predicts that infinite accelerations occur at a finite time, thus violating the validity of neglecting inertia. Stokes et al. [5] considered the extension under gravity of a mass of viscous fluid attached to the underside of a horizontal boundary. They considered both long-wavelength theory and finite element simulations and showed that, in the absence of inertia, infinite accelerations will also occur. Stokes et al. [9] included the effects of surface tension for a thread extending under gravity and paid particular attention to the role played by surface tension near the bottom end of the thread. Al Khatib [11] used long-wavelength theory to study a viscoplastic thread extending under gravity. Huang et al. [6] considered the stretching of glass tubes under gravity with externally applied heating that significantly affects the viscosity of the fluid. Huang et al. [7] generalized this approach to different types of extensional forces and derived explicit solutions that could be used to identify the factors that control the final shape of glass microelectrodes. Kaye [4] considered extensional flows with inertia, but did not discuss the role of inertia in cases in which the accelerations become large.

Although the above papers represent significant advances in understanding extensional flows, none of them addresses the mechanism by which inertia ultimately must become important. Wilson [3] argued that inertia will prevent the infinite accelerations that occur if inertia is completely neglected. He also argued that the minimum cross-sectional area will tend to zero as time tends to infinity if surface tension is completely neglected. However, inertial effects were not the main focus of Wilson's paper and the majority of the analysis is for the zero inertia case. The role of inertia in preventing infinite accelerations is also carefully discussed in Eggers & Villermaux [10]. In order to further address the question of inertial effects, Stokes & Tuck [12] used numerical methods to solve the full initial–boundary-value problem with inertia. They obtained numerical solutions of the long-wavelength equations and compared them with finite element simulations of the full Navier–Stokes equations for zero surface tension. They showed that the thread extends indefinitely, but that computing accurate numerical solutions is very challenging due to large gradients that develop in the solution. Bradshaw-Hajek et al. (2007) developed a more accurate numerical method for the long-wavelength equations and discussed the various difficulties in computing accurate solutions for large times. In addition, these numerical difficulties make determining when and if surface tension effects become important an extremely challenging proposition. In light of these numerical difficulties, an asymptotic solution to this problem would certainly be valuable and this provides one of the major motivations of this paper. We also note that Balmforth et al. [13] obtained numerical solutions for the dripping of a viscoplastic thread with inertial effects and found good agreement with laboratory experiments. The effects of inertia and viscous heating were considered by Wylie and Huang [14].

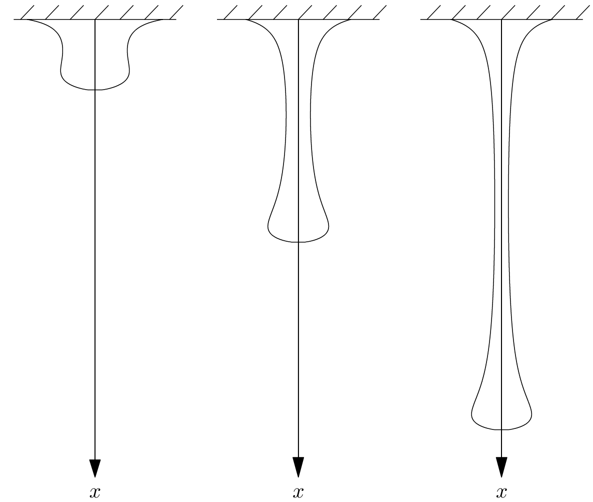

Fig. 1. Schematic of a viscous thread extending under gravity. The left panel represents the initial shape of the thread. The other two panels show the thread at two later times.

In fact, despite the extensive numerical work that has been done on this problem, there has been little analytical progress when inertial effects are included. In a previous paper, Wylie et al. [15] derived asymptotic solutions for the problem of a viscous thread that is pulled at its ends by an imposed fixed force. They showed that the solutions contain complicated internal layers. In this paper, we use an analogous approach and show that internal layers similar to those obtained by Wylie et al. [15] can occur. In addition, we show that the solutions for a thread extending under gravity exhibit new types of internal and boundary layers that were not present in the previous paper. These new layers require a number of novel asymptotic techniques and the structure of the solution can be surprisingly complicated. We use this solution to explain the mechanism that underlies the persistence and stability of long viscous threads.

In Section 2, we formulate the long-wavelength theory and derive a particularly convenient Lagrangian form of the governing equations. In Section 3, we completely neglect inertia and determine the explicit solution. We show that inertia must ultimately become important in one of two different ways. In the first case, inertia first becomes important at a location away from the upper boundary. In the second case, inertia first becomes important at the upper boundary. In Sections 4 and 5, we include the effects of inertia and determine the asymptotic solution for the first and second cases, respectively. Finally, we summarize our results and discuss the role of surface tension in Section 6.

2. Formulation

We consider a slender axisymmetric viscous thread that is suspended from a horizontal upper surface (which we will refer to as the 'boundary') and extends vertically downwards due to gravity (see Fig. 1). We assume that the thread is composed of an incompressible Newtonian fluid with density ρ, surface tension coefficient γ, and viscosity μ. We denote time by t, the vertical distance measured downwards from the boundary by x, the cross-sectional area of the thread by $S(x, t)$, the initial cross-sectional area by $S_0(x)$, the initial length of the thread by L, and the gravitational acceleration by g. We define the aspect ratio to be $\delta = \sqrt{\langle S_0\rangle}/L$, where

$$\langle S_0\rangle = \frac{1}{L}\int_0^L S_0(x)\mathrm{d}x. \tag{1}$$