WAVE SCATTERING BY MANY SMALL BODIES: TRANSMISSION BOUNDARY CONDITIONS

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Wave scattering by many ($M = M(a)$) small bodies, at the boundary of which transmission boundary conditions are imposed, is studied.

Smallness of the bodies means that $ka \ll 1$, where a is the characteristic dimension of the body and $k = 2\pi/\lambda$ is the wave number in the medium in which small bodies are embedded.

Explicit asymptotic formula is derived for the field scattered by a single small scatterer of an arbitrary shape. Equation for the effective field is derived in the limit as $a \to 0$ while $M(a) \to \infty$ at a suitable rate.

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1. Introduction

There is a large literature on "homogenization", which deals with the properties of the medium in which other materials are distributed. Quite often it is assumed that the medium is periodic, and homogenization is considered in the framework of G-convergence ([4, 5]). In most cases, one considers elliptic or parabolic problems with elliptic operators positive-difinite and having discrete spectrum.

The author has developed a theory of wave scattering by many small particles embedded in an inhomogeneous medium ([8–13]). One of the practically important consequences of his theory was a derivation of the equation for the effective (self-consistent) field in the limiting medium, obtained in the limit $a \to 0$, $M = M(a) \to \infty$, where a is the characteristic size of a small particle, and $M(a)$ is the total number of the embedded particles.

The theory was developed in [8–13] for boundary conditions (bc) on the surfaces of small bodies, which include the Dirichlet bc, $u|_{S_m} = 0$, where S_m is the surface of the m-th particle D_m, the impedance bc, $\zeta_m u|_{S_m} = u_N|_{S_m}$, where N is the unit normal to S_m, pointing out of D_m, ζ_m is the boundary impedance, and the Neumann bc, $u_N|_{S_m} = 0$.

The novelty in this paper is the development of a similar theory for the *transmission* (*interface*) bc:

$$\rho_m u_N^+ = u_N^-, \qquad u^+ = u^- \qquad \text{on } S_m, \quad 1 \le m \le M. \tag{1}$$

Here ρ_m is a constant, $+(-)$ denotes the limit of $\partial u/\partial N$, from inside (outside) of D_m.

The physical meaning of the transmission boundary conditions is the continuity of the pressure and the normal component of the velocity across the boundaries of the discontinuity of the density. One may think about problem (1)–(5) as of the problem of acoustic wave scattering by many small bodies.

The essential novelty of the theory, developed in this paper, is the asymptotically exact, as $a \to 0$, treatment of the one-body and many-body scalar wave scattering problem in the case of small scatterers on the boundaries of which the transmission boundary conditions are imposed. An analytic explicit asymptotic formula for the field scattered by one small body is derived. An integral equation for the limiting effective field in the medium, in which many small bodies are embedded, is derived in the limit $a \to 0$ and $M(a) \to \infty$, where $M(a)$ is the total number of the embedded small bodies (particles), and $M = M(a)$ tends to infinity at a suitable rate as $a \to 0$.

For the problem with the number M of particles not large, say, less than 5000, our theory gives an efficient numerical method for solving many-body wave scattering problem.

For the problem with M very large, say, larger than 10^5, the solution to many-body wave scattering problem consists in numerical solution of the integral equation for the limiting field in the medium, in which small particles are embedded. The solution to this equation approximates the solution to the many-body wave scattering problem with high accuracy.

Our approach is quite different from the approach developed in homogenization theory, we do not assume periodicity in the location of the small scatterers. Our results are of interest also in the case when the number of scatterers is not large, so the homogenization theory is not applicable.

Let us formulate the scattering problem we are treating,

$$\text{Let } \Omega := \bigcup_{m=1}^{M} D_m, \qquad \Omega' = \mathbb{R}^3 \backslash \Omega,$$

$$(\nabla^2 + k^2) u = 0 \qquad \text{in } \Omega', \tag{2}$$

$$(\nabla^2 + k_m^2) u = 0 \qquad \text{in } D_m, \qquad 1 \le m \le M, \tag{3}$$

$$u = u_0 + v, u_0 = e^{ik\alpha\cdot x}, \qquad \alpha \in S^2, S^2 \text{ is a unit sphere in } \mathbb{R}^3, \tag{4}$$

$$r\left(\frac{\partial v}{\partial r} - ikv\right) = o(1), \qquad r \to \infty. \tag{5}$$

We assume that ρ_m, k and k_m^2 are fixed given positive constants, and the surfaces