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Stability of front solutions in a model for a surfactant driven flow on an inclined plane

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HIGHLIGHTS

- We study fronts in a model for a flow of a thin liquid film down an inclined plane.
- In a system without a surfactant, the front has no unstable spectrum.
- With a surfactant, there are regimes when the front has no unstable spectrum.
- The results are based on analytical and numerical methods.

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ABSTRACT

We consider a model for the flow of a thin liquid film down an inclined plane in the presence of a surfactant. The model is known to possess various families of traveling wave solutions. We use a combination of analytical and numerical methods to study the stability of the traveling waves. We show that for at least some of these waves the spectra of the linearization of the system about them are within the closed left-half complex plane.

1. Introduction

In this paper we analyze a model that describes the flow of thin liquid film down the inclined plane, which is modified by the presence of insoluble surfactant, and with the effects of gravity taken into account [1,2]. Surfactants as media that stays on the surface of a thin film flow have a variety of applications, from industrial to medical [3–6]. Rooted in lubrication theory [1,2], the model consists of a system of nonlinearly coupled partial differential equations

$$
\begin{aligned}
&h_t - \frac{1}{2}\left(h^2 \Gamma_x\right)_x + \frac{\alpha}{3}\left(h^3\right)_x = \frac{\beta}{3}\left(h^3 h_x\right)_x,\\
&\Gamma_t - \left(h\Gamma\Gamma_x\right)_x + \frac{\alpha}{2}\left(h^2\Gamma\right)_x = \frac{\beta}{2}\left(h^2 h_x \Gamma\right)_x + D\Gamma_{xx},
\end{aligned}
\tag{1.1}
$$

where $h(x,t)$ represents the height of the thin film, $\Gamma(x,t)$ represents the surfactant concentration at time t and x is the space variable along the inclined plane. Parameter D is proportional to the inverse of the Péclet number and acts as a diffusion constant of the surfactant concentration. Péclet number measures the relative contribution of mass transport by diffusion against mass transport by advection. Smaller values of D, which we assume positive, indicate the larger influence of advection, larger ones indicate the stronger influence of the diffusion. Parameters α and β encode the steepness of the incline: α is proportional to the sine of the angle formed by the inclined plane with the horizontal surface, and β is

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proportional to the cosine of the same angle. This model takes into account the Marangoni force, which is due to the modification of the surface tension by the presence of a surfactant.

The system (1.1) is a limiting system for

$$\begin{aligned} & h_t - \frac{1}{2}\left(h^2\Gamma_x\right)_x + \frac{\alpha}{3}\left(h^3\right)_x = -\frac{C}{3}\left(h^3 h_{xxx}\right)_x + \frac{\beta}{3}\left(h^3 h_x\right)_x, \\ & \Gamma_t - (h\Gamma\Gamma_x)_x + \frac{\alpha}{2}\left(h^2\Gamma\right)_x \\ & \quad = -\frac{C}{2}\left(h^2 h_{xxx}\Gamma\right)_x + \frac{\beta}{2}\left(h^2 h_x \Gamma\right)_x + D\Gamma_{xx}, \end{aligned} \tag{1.2}$$

as $C \to 0$, where C is a quantity proportional to the capillary number. The system (1.2) is obtained from the two-dimensional Navier–Stokes equation in [7,8,6,2] and the existence of traveling wave solutions that connect two different constant states have been studied in [9–11]. In [9,10], the authors consider the case where at least one of the parameters is zero, while [11] studies different parameter regimes when all the parameters are positive.

The stability of traveling waves very often is of critical importance. For example, in surfactant replacement therapy a coating of surfactant is used to support a healthy lung function [12]. The development of stable wavefronts is related to the mechanism of the surfactant delivery for the lung.

Numerical simulations performed in [10] suggest that in some parameter regimes traveling wave solutions are stable. It is also mentioned in [10] that the analysis of the stability of the individual waves of (1.2) when all parameters are nonzero is of interest for applications. In this paper we study the stability of wavefronts of the limiting for (1.2) system (1.1). The stability of these wavefronts will factor in the stability analysis of wavefronts in the full system (1.2) through the multi-scale approach. The latter will be a subject of future work.

We note that the parameter α can be normalized to be 1 in the above equations by rescaling

$$\bar{x} = \alpha x, \qquad \bar{t} = \alpha^2 t,$$

and dropping the bars as in [11]. After this transformation, the system (1.1) reads

$$\begin{aligned} & h_t - \frac{1}{2}\left(h^2\Gamma_x\right)_x + \frac{1}{3}\left(h^3\right)_x = \frac{\beta}{3}\left(h^3 h_x\right)_x, \\ & \Gamma_t - (h\Gamma\Gamma_x)_x + \frac{1}{2}\left(h^2\Gamma\right)_x = \frac{\beta}{2}\left(h^2 h_x\Gamma\right)_x + D\Gamma_{xx}. \end{aligned} \tag{1.3}$$

Traveling waves are sought as stationary solutions of the form $(h(\xi), \Gamma(\xi))$, where $\xi = x - st$ is the traveling wave coordinate, and s is an undetermined at the moment parameter related to the speed of the wave, so $(h(\xi), \Gamma(\xi))$ solves the following system of ordinary differential equations

$$\begin{aligned} & sh_\xi - \frac{1}{2}\left(h^2\Gamma_\xi\right)_\xi + \frac{1}{3}\left(h^3\right)_\xi = \frac{\beta}{3}\left(h^3 h_\xi\right)_\xi, \\ & s\Gamma_\xi - (h\Gamma\Gamma_x)_\xi + \frac{1}{2}\left(h^2\Gamma\right)_\xi = \frac{\beta}{2}\left(h^2 h_\xi\Gamma\right)_\xi + D\Gamma_{\xi\xi}. \end{aligned} \tag{1.4}$$

To capture wavefronts in this system one imposes boundary-like conditions

$$h(-\infty) = h_L, \qquad h(+\infty) = h_R, \qquad \Gamma(\pm\infty) = 0, \tag{1.5}$$

which indicate that we are interested in traveling waves that are shaped as a front in the h-component and as a pulse in Γ-component. With these boundary condition, integration of (1.4) leads to

$$\begin{aligned} & -hs - \frac{1}{2}h^2\Gamma' + \frac{1}{3}h^3 = \frac{\beta}{3}h^3 h' + K_1, \\ & -\Gamma s - h\Gamma\Gamma' + \frac{1}{2}h^2\Gamma = \frac{\beta}{2}h^2 h'\Gamma + D\Gamma', \end{aligned} \tag{1.6}$$

where the derivative is taken with respect to ξ, and the quantity

$$K_1 = -\frac{1}{3}h_L h_R(h_L + h_R) \tag{1.7}$$

is the constant of integration expressed through h_L and h_R.

The wave speed is related to the boundary values as

$$s = \frac{1}{3}(h_L^2 + h_L h_R + h_R^2). \tag{1.8}$$

The system (1.6) can be simplified by replacing the second equation with a linear combination of the first and second equations with respective coefficients $-\frac{2}{h}$ and $\frac{2}{\Gamma}$ and written as a first order system

$$\begin{aligned} & h' = \frac{1}{\beta h^3}\left(h^3 - 3sh - 3K_1 - 3h\Gamma\frac{sh + 3K_1}{h\Gamma + 4D}\right), \\ & \Gamma' = \frac{2\Gamma}{h}\frac{sh + 3K_1}{h\Gamma + 4D}. \end{aligned} \tag{1.9}$$

We assume as in [10] that

$$0 < \frac{h_R}{h_L} < \frac{\sqrt{3} - 1}{2}, \tag{1.10}$$

where h_L and h_R are the positive roots of the polynomial $h^3 - 3sh - 3K_1$, which can be factored as

$$h^3 - 3sh - 3K_1 = (h - h_L)(h - h_R)(h + h_L + h_R). \tag{1.11}$$

Then, the traveling wave solutions correspond to the heteroclinic connections that asymptotically connect $(h, \Gamma) = (h_R, 0)$ to $(h, \Gamma) = (h_L, 0)$. An example of such heteroclinic connection, together with the traveling wave it represents, is shown in Fig. 4.1. The existence of continuum of such solutions (parametrized by the maximum value of Γ) is shown in [11] (more specifically, see sections called Region 2, Region 4, and Region 5 in [11]). In [11], the system (1.9) represents the reduced flow on a normally hyperbolic invariant manifold description of which is obtained by exploiting the multi-scale structure of the full system (1.2) when the parameter C is small.

We point out that the line $\Gamma = 0$ is an invariant set for the flow induced by the system (1.9), and there is an asymptotic connection between $(h_L, 0)$ to $(h_R, 0)$ along the set $\Gamma = 0$.

The main ingredient of the stability analysis of a traveling wave is to find the location of the spectrum of the linearization of the pde system (1.3) about the traveling wave. The presence of spectra with positive real parts indicates an instability as perturbations to the wave then grow in amplitude at exponential rates. Spectrum on the imaginary axis indicates that the perturbations may not decay. In this paper, in Section 2 we analytically prove that the linearization of the pde system (1.3) about the wavefront with $\Gamma = 0$ does not have spectrum with nonnegative real parts, with the exception of the origin. We also perform Evans function numerical computations (Section 4) combined with energy estimates (Section 3) which show that some of the wavefronts with $\Gamma \neq 0$ do not have spectrum within the closed right half of complex plane, with the exception of the spectrum at the origin.

Let (h, Γ) be traveling wave solutions of (1.6). The linearization of the pde system (1.3) around the traveling wave solutions (h, Γ) gives rise to the following eigenvalue problem

$$\begin{aligned} \lambda V &= \left(sV + \frac{1}{2}(2h\Gamma_\xi V + h^2 U_\xi) - h^2 V \right.\\ &\quad \left. + \frac{\beta}{3}(3h^2 h_\xi V + h^3 V_\xi)\right)_\xi, \\ \lambda U &= \left(sU + \Gamma\Gamma_\xi V + h\Gamma_\xi U + h\Gamma U_\xi - h\Gamma V - \frac{1}{2}h^2 U \right.\\ &\quad \left. + \frac{\beta}{2}(2h\Gamma h_\xi V + h^2 h_\xi U + h^2\Gamma V_\xi) + DU_\xi\right)_\xi. \end{aligned} \tag{1.12}$$