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Transitions between streamline topologies of structurally stable Hamiltonian flows in multiply connected domains

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HIGHLIGHTS

- A combinatorial procedure providing a list of possible transitions between streamline topologies.
- It is applicable to any physical phenomena described by 2D Hamiltonian vector fields.
- It provides many global and generic transitions between streamline topologies.
- It is applicable to snapshots of streamline patterns observed experimentally.
- A new data compression algorithm for a large amount of long-time flow evolution data.

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ABSTRACT

We consider Hamiltonian vector fields with a dipole singularity satisfying the slip boundary condition in two-dimensional multiply connected domains. One example of such Hamiltonian vector fields is an incompressible and inviscid flow in exterior multiply connected domains with a uniform flow, whose Hamiltonian is called the stream function. Here, we are concerned with topological structures of the level sets of the Hamiltonian, which we call streamlines by analogy from incompressible fluid flows. Classification of structurally stable streamline patterns has been considered in Yokoyama and Sakajo (2013), where a procedure to assign a unique sequence of words, called the maximal word, to these patterns is proposed. Thanks to this procedure, we can identify every streamline pattern with its representing sequence of words up to topological equivalence. In the present paper, based on the theory of word representations, we propose a combinatorial method to provide a list of possible transient structurally unstable streamline patterns between two different structurally stable patterns by simply comparing their maximal word representations without specifying any Hamiltonian. Although this method cannot deal with topological streamline changes induced by bifurcations, it reveals the existence of many non-trivial global transitions in a generic sense. We also demonstrate how the present theory is applied to fluid flow problems with vortex structures.

1. Introduction

Flow problems in two-dimensional multiply connected domains in the presence of a uniform flow are of significance from an applications point of view, since they are often regarded as mathematical models of biofluids and environmental flows such as the schooling of fish in rivers and coastal current flows in the presence of many islands. Suppose here that the flow is incompressible. Then the instantaneous velocity field $(u(t, x, y), v(t, x, y))$ at a location (x, y) and at time t is given by $u = \partial_y \psi$ and $v = -\partial_x \psi$ for a stream function $\psi(t, x, y)$, which gives rise to a Hamiltonian vector field with ψ being the Hamiltonian. In this paper, we are concerned with topological structures of level sets of the stream function, called streamlines, and their transitions. Streamline topology plays a significant role in characterizing incompressible flows and has been considered in many fluid problems. For instance, Brøns et al. [1] described the bifurcations of streamline topologies of a flow around a circular cylinder at moderate Reynolds numbers. Brøns and

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Hartnack [2] also investigated the bifurcations of streamline patterns of the steady Navier–Stokes, or Stokes flows, near some degenerate critical points away from boundaries. More examples are found in [3–7]. These studies are based on the local bifurcation analysis of the Hamiltonian associated with the fluid problems.

In the meantime, there is a combinatorial approach to investigate streamline topologies. For example, Aref and Brøns [8] have considered a classification of streamline topologies generated by potential flows with point-vortex singularities, called *vortex flows*, in the 2D unbounded plane. This study has been extended to the Hamiltonian vector fields, which are a generalization of the vortex flows, in 2D multiply connected domains [9,10], in which the classification of *structurally stable* streamline topologies that are unchanged under small perturbations are considered. In these studies, the Hamiltonian vector fields are assumed to satisfy the slip boundary condition. The classification theory also provides a procedure to assign a unique sequence of words to every streamline topology, which allows us to identify each structurally stable streamline pattern with this sequence of words up to topological equivalence. In addition, it is possible to describe an evolution of streamline topologies as a change of words by assigning the unique sequence of word to an instantaneous streamline patterns at each time of the evolution.

When a transition of streamline topologies is represented by a change of the words, it is natural to ask whether a transient streamline pattern that lies between these patterns can be determined from their word representations. This is not an easy task, since the transient streamline pattern is structurally unstable and it may thus contain many singular streamline structures [11]. In the present study, we propose a combinatorial method providing a list of possible transitions between two structurally stable streamline topologies from their word representations. It is also able to show which transitions are impossible by just comparing the word representations. The construction of this paper is as follows. In the next section, we review the theory of word representations for structurally stable Hamiltonian vector fields in multiply connected domains developed in [9,10]. We then make some preparations for describing transitions between structurally stable Hamiltonian vector fields. In particular, we introduce two special sets of structurally unstable Hamiltonian vector fields, called *h-unstable* and *p-unstable* vector fields. We use these unstable vector fields to describe transient streamline patterns, since it is mathematically shown in Section 3 that they are open dense in the set of structurally unstable vector fields under certain assumptions. In Sections 4 and 5, we introduce some operations that generate *h*-unstable and *p*-unstable streamline patterns from structurally stable patterns and observe how small perturbations of these unstable patterns give rise to changes of the structurally stable patterns and their word representations. Let us remark that, in contrast to the bifurcation analysis in the preceding studies, our mathematical approach is combinatorial and no mathematical analysis of the Hamiltonian is required. In Section 6, we explain how the present theory is applied to describing transitions of streamline topologies of fluid flows from their word representations. In Section 7, we provide some applications to incompressible fluid flows to figure out some features of the combinatorial method and differences from the preceding bifurcation analysis [1,7]. In the final section, we summarize the global transition analysis given in this paper and we then discuss about its significance.

2. Word representation for structurally stable Hamiltonian vector fields

The global transition analysis given in this paper is based on the theory of word representations for streamline topologies of structurally stable Hamiltonian vector fields satisfying the slip boundary condition in two-dimensional multiply connected domains with a dipole singularity [10], which is reviewed as follows. In order to characterize multiply connected domains topologically, we use the term *genus element* instead of *genus* usually used in mathematical studies. A genus element represents not only a physical obstacle and a singular point where the value of the Hamiltonian diverges, but also an elliptic fixed point of the vector field. This is an unconventional but necessary term, since elliptic fixed points can change to saddle fixed points owing to the bifurcation and then they are not regarded as genus. However, in the present paper, assuming that no bifurcation generating or removing elliptic fixed points occurs, we restrict our attention to the global transitions between structurally stable streamline topologies with the same number of genus elements. Therefore, it is unnecessary to distinguish physical obstacles, singular points and elliptic fixed points, and thus all genus elements are schematically represented as circular holes in the following illustrations. This means that the number of genus elements does not always coincide with that of physical obstacles in the domain. In other words, any circular obstacle without saddle points at its boundary can be replaced by a singular point or an elliptic fixed point. Let us remark that the bifurcation of fixed points is an interesting phenomenon mathematically as well as physically even if it is prohibited in this paper. We will show some examples in Section 7 to see the difference between the changes of topological streamline patterns handled by the present theory and those by the bifurcation analysis, and we then discuss more about it in the last section.

Since the uniform flow exists in unbounded domains, we need to consider an exterior domain with M genus elements in the complex z-plane, which is denoted by $\mathcal{D}_z(M)$. Since all genus elements are represented as circular holes in this paper, the domain has $M+1$ circular boundaries. In what follows for a technical reason, we consider the topological streamline structures of Hamiltonian vector fields in a multiply connected bounded domain $\mathcal{D}_\zeta(M)$ with the same number of genus elements in the complex ζ-plane by constructing the conformal mapping from $\mathcal{D}_z(M)$ to $\mathcal{D}_\zeta(M)$. This causes no serious problem, since the streamline topologies are invariant under the action of the conformal mapping. Then the uniform flow is characterized in terms of a Hamiltonian vector field in the bounded domain as follows. Since the uniform flow is irrotational, it is represented by a complex potential $W_U(z)$ that behaves like $W_U(z) \sim U\mathrm{e}^{-i\phi}z$ as $z \to \infty$ for the flux U and the angle of inclination to the real axis ϕ in the exterior domain $\mathcal{D}_z(M)$. For a conformal mapping $z = f(\zeta)$ from the bounded multiply connected $\mathcal{D}_\zeta(M)$ to $\mathcal{D}_z(M)$ with $f(0) = \infty$, the asymptotic behavior of the conformal mapping in the neighborhood of the origin becomes $z = f(\zeta) \sim \frac{a}{\zeta}$ for some constant a. Hence, the complex potential of the uniform flow is given by $W_U(\zeta) \sim U\mathrm{e}^{-i\phi}\frac{a}{\zeta}$ as $\zeta \to 0$. Here we may set $U = 1$, $\phi = 0$ and $a = 1$ without loss of generality, since we are interested in the topological streamline structure of the uniform flow in the neighborhood of the origin. We call the singular point at the origin of $\mathcal{D}_\zeta(M)$ the *1-source–sink point* whose definition is given as follows [10].

Definition 2.1. A point $p \in \mathcal{D}_\zeta(M)$ is said to be a 1-source–sink point, if $V|_{\mathcal{D}_\zeta(M)\setminus\{p\}}$ is a vector field on $\mathcal{D}_\zeta(M) \setminus \{p\}$ generated by a stream function ψ, for which there is a pair of a neighborhood U of p and a homeomorphism h from U to the unit disk D with $h(p) = 0$ such that $\psi \circ h^{-1}|_{D\setminus\{0\}} = -\frac{\sin\theta}{r}$ in the polar coordinates (r, θ) associated with the disk D.

The 1-source–sink point is a mathematical expression for a uniform flow in $\mathcal{D}_\zeta(M)$ in terms of Hamiltonian vector fields and it induces a dipole-like singular streamline pattern locally in the neighborhood of the origin as shown in Fig. 1(a), which consists of infinitely many self-connecting orbits to the 1-source–sink point.