WHEN QUANTUM CHANNEL PRESERVES PRODUCT STATES*

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Product states are always considered as the states that do not contain quantum correlation. We discuss here when a quantum channel sends the product states to themselves. Exact forms of such channels are proposed. It is shown that such a quantum channel is a local quantum channel, a composition of a local quantum channel and a flip operation, or such that one of the local states is fixed. Both finite- and infinite-dimensional systems are considered.

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Quantum systems can be correlated in the ways inaccessible to classical objects. This quantum feature of correlations not only is the key to our understanding of quantum world, but also is essential for the powerful applications of quantum information and quantum computation. Product state is the state without any quantum correlation [1, 2]. It is the only state that has zero mutual information [3] which is interpreted as a measure of total correlations between its two subsystems. It neither contains quantum discord (QD) [4] nor contains the measurement-induced nonlocality (MIN) [5, 6]. Recently, it has been shown that the super discord [7] of ρ_{ab} is zero if and only if it is a product state [8].

In particular, it is crucial to study the behaviour of quantum correlation under the influence of noisy channel [6, 9–21]. For example, local channel that cannot create QD is investigated in [9, 18, 19], local channel that preserves the state with vanishing MIN is characterized in [6] and local channel that preserves the maximally entangled states is explored in [20]. The goal of this paper is to discuss when a quantum channel preserves the product states.

We fix some notation first. Let H, K be separable complex Hilbert spaces and $\mathcal{B}(H, K)$ ($\mathcal{B}(H)$ when $K = H$) be the Banach space of all (bounded linear) operators from H into K. Recall that $A \in \mathcal{B}(H)$ is self-adjoint if $A = A^\dagger$ ($A^\dagger$ stands for the adjoint operator of A); and A is positive, denoted by $A \geq 0$, if A is self-adjoint with the spectrum falling in the interval $[0, +\infty)$ (or equivalently, $\langle\psi|A|\psi\rangle \geq 0$ for all

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$|\psi\rangle \in H$). A linear map $\phi: \mathcal{B}(H) \to \mathcal{B}(K)$ is called a positive map if $A \geq 0$ implies $\phi(A) \geq 0$ for any $A \in \mathcal{B}(H)$. Let $\mathcal{M}_n(\mathcal{B}(H))$ be the algebra of all n by n matrices with entries being operators in $\mathcal{B}(H)$. Let $\mathbb{1}_n \otimes \phi : \mathcal{M}_n(\mathcal{B}(H)) \to \mathcal{M}_n(\mathcal{B}(K))$ be the map defined by $(\mathbb{1}_n \otimes \phi)[X_{ij}] = [\phi(X_{ij})]$. We say that ϕ is completely positive if $\mathbb{1}_n \otimes \phi$ is positive for any n.

We review the definition of the quantum channel. Let $\mathcal{T}(H)$ and $\mathcal{T}(K)$ be the trace classes on H and K, respectively. Recall that a quantum channel is described by a trace-preserving completely positive linear map $\phi: \mathcal{T}(H) \to \mathcal{T}(K)$. Every quantum channel ϕ between two systems, associated respectively with Hilbert spaces H and K, admits the form [22]

$$\phi(\cdot) = \sum_i X_i(\cdot)X_i^\dagger, \tag{1}$$

where $\{X_i\} \subset \mathcal{B}(H, K)$ satisfy $\sum_i X_i^\dagger X_i = I_H$, I_H is the identity operator on H. If $\dim H = +\infty$ and $\dim K = +\infty$, then there may have infinite X_is in Eq. (1). We call ϕ a *completely contractive channel* if $\phi(\mathcal{S}(H))$ is a single state [23], i.e. there exists a fixed state $\omega_0 \in \mathcal{S}(H)$ such that

$$\phi(\cdot) = \mathrm{Tr}(\cdot)\omega_0. \tag{2}$$

Let $H_{ab} = H_a \otimes H_b$ with $\dim H_a \leq +\infty$ and $\dim H_b \leq +\infty$ be the state space of a bipartite system A + B. Let $\mathcal{S}(H_{ab})$ and $\mathcal{S}_P(H_{ab})$ be the set of all quantum states acting on H_{ab} and the set of all product states in $\mathcal{S}(H_{ab})$, respectively. That is $\mathcal{S}_P(H_{ab}) = \{\rho \otimes \delta : \rho \in \mathcal{S}(H_a), \delta \in \mathcal{S}(H_b)\}$. Let $\{|i\rangle\}$ and $|j'\rangle$ be the orthonormal bases of H_a and H_b, respectively. The operator $F = \sum_{i,j} |i\rangle|j'\rangle\langle j'|\langle i|$ is called the swap operator from $H_{ba} = H_b \otimes H_a$ to H_{ab} [24], namely, $F|\psi_b\rangle|\psi_a\rangle = |\psi_a\rangle|\psi_b\rangle$ for any $|\psi_b\rangle|\psi_a\rangle \in H_{ba}$ (note that F is an isometry since $F^\dagger F = I_{ba}$). Then $F\delta \otimes \rho F = \rho \otimes \delta \in \mathcal{S}_P(H_{ab})$ for any $\delta \otimes \rho \in \mathcal{S}_P(H_{ba})$. We denote it by $f(\delta \otimes \rho) := F\delta \otimes \rho F$.

The following is the main result of this paper.

THEOREM 1. *Let $\phi : \mathcal{T}(H_{ab}) \to \mathcal{T}(H_{ab})$ be a quantum channel. Then $\phi(\mathcal{S}_P(H_{ab})) \subseteq \mathcal{S}_P(H_{ab})$ if and only if it has one of the following forms.*

(i) *$\phi = \phi_a \otimes \phi_b$, where ϕ_a and ϕ_b denote the local quantum channels on parts A and B respectively;*

(ii) *$\phi = f \circ (\psi_a \otimes \psi_b)$, where ψ_a is a quantum channel from $\mathcal{T}(H_a)$ to $\mathcal{T}(H_b)$ and ψ_b is a quantum channel from $\mathcal{T}(H_b)$ to $\mathcal{T}(H_a)$;*

(iii) *$\phi(\cdot) = \sigma \otimes \Lambda_b(\cdot)$, where σ is a state of part A, Λ_b is a quantum channel from $\mathcal{T}(H_{ab})$ to $\mathcal{T}(H_b)$;*

(iv) *$\phi(\cdot) = \Lambda_a(\cdot) \otimes \tau$, where τ is a state of part B, Λ_a is a quantum channel from $\mathcal{T}(H_{ab})$ to $\mathcal{T}(H_a)$.*

Theorem 1 implies that a quantum channel sends product states to product states if and only if it is an action of two local operations on the parts A and B, respectively, or is an action of quantum channel from the total system to a subsystem with another reduced state fixed. (In Theorem 1, for the notation $\sigma \otimes \Lambda_b(\cdot)$ and $\Lambda_a(\cdot) \otimes \tau$, with some abuse of terminology, σ can be viewed as a completely