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ON DIFFERENTIAL FORM METHOD TO FIND LIE SYMMETRIES OF TWO TYPES OF TODA LATTICES

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In this paper, we investigate Lie symmetries of the $(1+1)$-dimensional celebrated Toda lattice and the $(2+1)$-dimensional modified semidiscrete Toda lattice by using the extended Harrison and Estabrook's geometric approach. Two closed ideals written in terms of a set of differential forms are constructed for Toda lattices. Moreover, commutation relations of a Kac–Moody–Virasoro type Lie algebra are obtained by direct computation.

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1. Introduction

Nowadays it is important to investigate symmetry properties and construct exact solutions of nonlinear differential-difference equations (DDEs). The concept of conditional symmetries is extended from differential equations to DDEs. Moreover, some well-known methods of investigating conditional symmetries have been developed, such as the intrinsic method [1], the generalized conditional symmetry method [2], and the extended CK's direct method [3, 4].

Most calculations of symmetries are performed by using the classical method for solving DDEs. In 1971, Harrison and Estabrook proposed a geometrical method to find symmetries of differential equations using a differential form technique [5–7]. The theory of the differential form method was extensively developed by Edelen [8, 9], and some computer programs were made in the REDUCE [10]. More recently, by using the discrete exterior differential technique, the method proposed by Harrison and Estabrook was extended to study the $(2+1)$-dimensional Toda equation and discrete time Toda equation, respectively [11–15]. In particular, Damianou and Sophocleous [16] examined interrelation among Noether symmetries, master symmetries and recursion operators for the Toda lattice, and investigated also invariants, higher Poisson brackets etc. For the case of two degrees of freedom, they proved that the Toda lattice is super-integrable. In this paper, we study Lie symmetries of the $(1+1)$-dimensional celebrated lattice and the $(2+1)$-dimensional modified semidiscrete Toda lattice using the extended Harrison and Estabrook's differential form method.

The rest of the paper is organized as follows. In Section 2, we present a short summary of the exterior differential calculus. In Section 3, the Lie symmetries of a $(1+1)$-dimensional celebrated Toda lattice are obtained by the extended method. In Section 4, the method is further used to investigate a $(2+1)$-dimensional modified semidiscrete Toda lattice. Finally, some conclusions and discussions are presented in the last section.

2. Difference and differential forms

Following closely Harrison and Estabrook, this section starts with a short summary of the exterior differential calculus that will be useful in the rest of this paper. For a more detailed description we refer the reader to [5, 11–13].

Let us assume that L is a lattice and A is an algebra of complex-valued functions on L. The right and left shift operators $E_\lambda, E_\lambda^{-1}$ at a node $x \in L$ in the λ-direction

$$E_\lambda x = x + \hat{\lambda}, \qquad E_\lambda^{-1} x = x - \hat{\lambda}, \tag{2.1}$$

define a homeomorphism on the function space A. That is to say

$$E_\lambda f(x) = f(x+\hat{\lambda}), \quad E_\lambda(f(x)\cdot g(x)) = E_\lambda f(x)\cdot E_\lambda g(x), \quad f, g \in A, \tag{2.2}$$

where $\hat{\lambda}$ is the spacing in the λ-direction and the dot denotes multiplication in A.

Next, the tangent space can be defined at the node x of L as $T_x L :=$ span$\{\Delta_\lambda|_x, \lambda = 1, 2, \ldots, n\}$. Here Δ_λ is the difference in the λ-direction defined by

$$\Delta_\lambda f(x) := (E_\lambda - \mathrm{Id})f = f(x+\hat{\lambda}) - f(x), \tag{2.3}$$

where Id is an identity mapping.

The dual space $T_x^* L$ of $T_x L$ is the space of 1-forms with a basis χ^λ defined on the link between x and $(x+\hat{\lambda})$. The base between $T_x L$ and $T_x^* L$ satisfy

$$\chi^\lambda(\Delta_\nu) = \delta_\nu^\lambda, \qquad \lambda, \nu = 1, 2, \ldots, n, \tag{2.4}$$

where δ is a Kronecker function.

Let us introduce the tangent bundle and its dual over L,

$$T(L) := \bigcup_{x\in L} T_x L, \qquad T^*(L) := \bigcup_{x\in L} T_x^* L, \tag{2.5}$$

respectively.

Then one can define the vectors in $T(L)$ and construct the whole differential algebra $\Omega^* = \bigoplus_{n\in Z} \Omega^n$ on $T^*(L)$, where Ω^n is the set of n-forms. The exterior derivative operator $d : \Omega^k \to \Omega^{k+1}$ is defined as

$$d\omega = \Delta_\alpha f \chi^\alpha \wedge \chi^{\lambda_1} \wedge \cdots \wedge \chi^{\lambda_k} \in \Omega^{k+1}, \tag{2.6}$$

where $\omega = f\chi^{\lambda_1} \wedge \cdots \wedge \chi^{\lambda_k} \in \Omega^k$ and $\alpha = 1, 2, \ldots$.